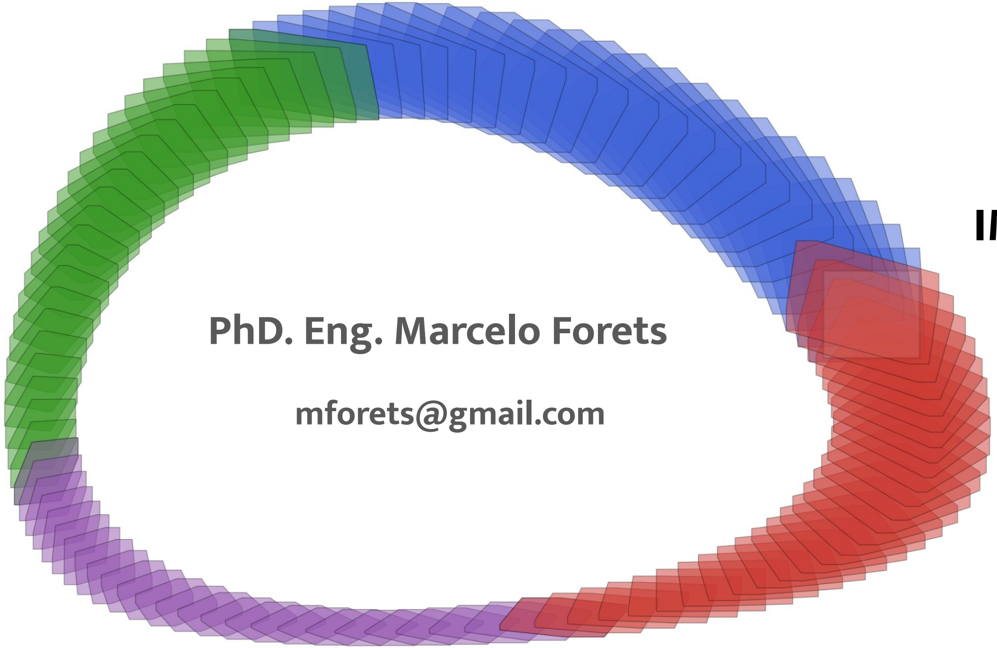
Material disponible en: https://github.com/mforets/PresentacionIMERL

Una introducción a la integración numérica basada en conjuntos



PhD. Eng. Marcelo Forets

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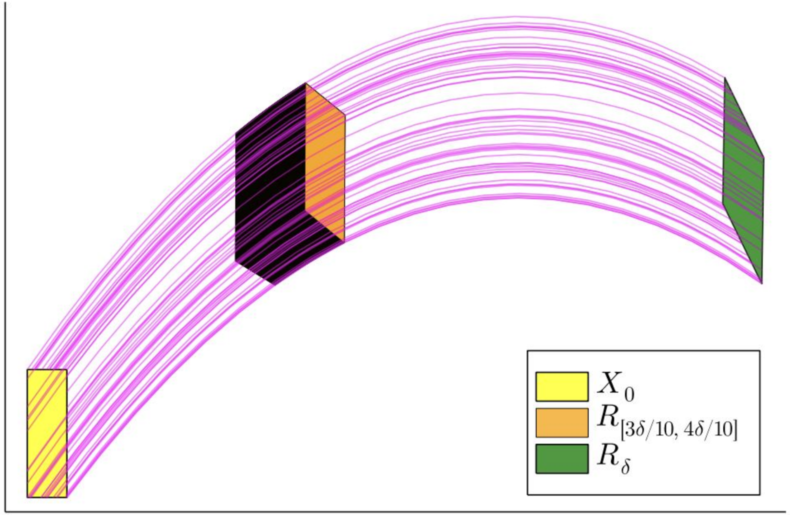
Seminario de EDPs y afines.
IMERL, Facultad de Ingeniería, UdelaR.
Septiembre 2024

UNIVERSIDAD DE LA REPÚBLICA URUGUAY

CURE Centro Universitario Regional del Este

Formulación del problema de reachability

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{w}(t)), \ \mathbf{x}(t) \in \mathbb{R}^n, \ \mathbf{w}(t) \in \mathcal{W}$$

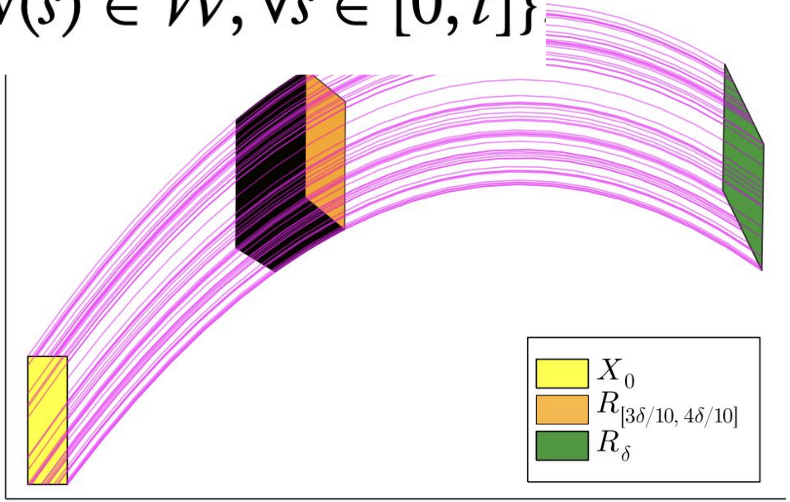


(a) Sketch of the reachability problem.

Formulación del problema de reachability

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{w}(t)),\ \mathbf{x}(t) \in \mathbb{R}^n,\ \mathbf{w}(t) \in \mathcal{W},$$

$$\mathrm{Reach}_t(\mathcal{X}_0) = \{\xi(t, x_0, \mathbf{w}) \mid x_0 \in \mathcal{X}_0,\ \mathbf{w}(s) \in \mathcal{W}, \forall s \in [0, t]\}.$$



(a) Sketch of the reachability problem.

Formulación del problema de reachability

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{w}(t)),\ \mathbf{x}(t) \in \mathbb{R}^n,\ \mathbf{w}(t) \in \mathcal{W},$$

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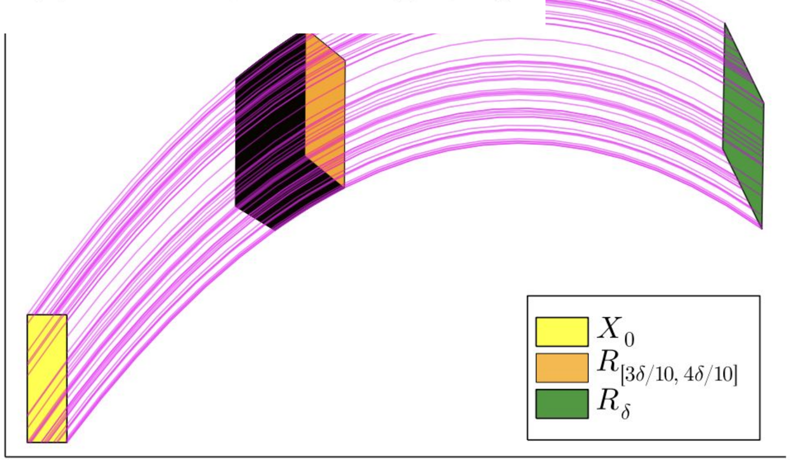
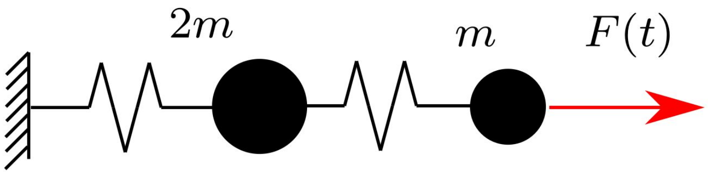
$$\boxed{\mathsf{Reach}_{[0,T]}(\mathcal{X}_0) = \bigcup_{t \in [0,T]} \mathsf{Reach}_t(\mathcal{X}_0)}$$



(a) Sketch of the reachability problem.

Ejemplo

Sistema masa-resorte acoplado



$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = F(t)$$

where

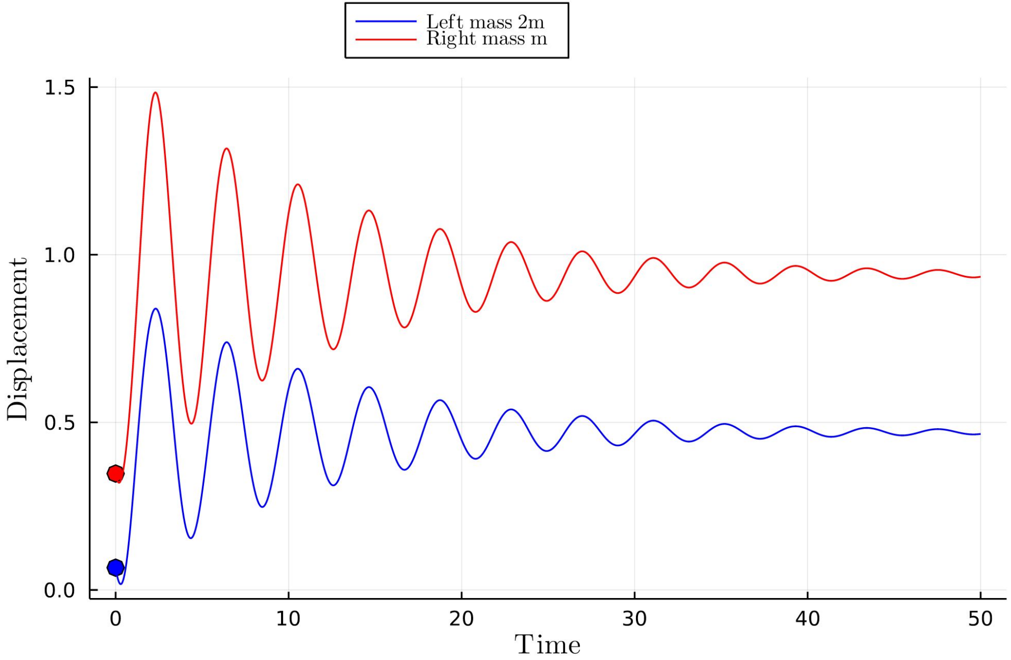
$$M = \begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix}, \quad K = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}, \quad C = \frac{M+K}{20},$$

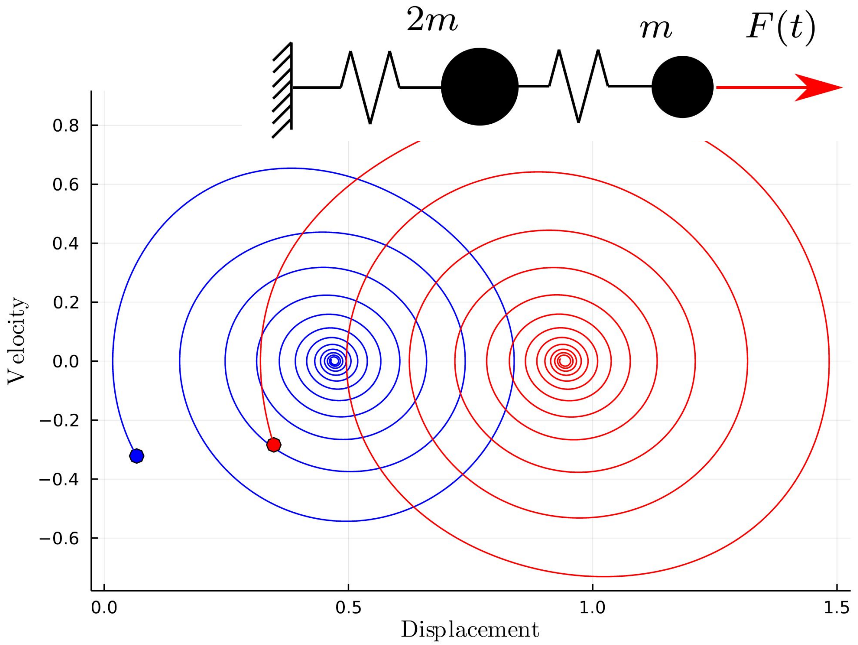
$$m = \frac{1}{4}, \quad k = 2, \quad F = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Github: https://github.com/JuliaReach/SetPropagation-FEM-JuliaCon21

Forets, M., & Caporale, D. F. (2022, June). Computing Reachable Sets of Semi-Discrete Solid Dynamics Equations with ReachabilityAnalysis.jl. In Proceedings of the JuliaCon Conferences (Vol. 1, No. 1, p. 95).

Ejemplo





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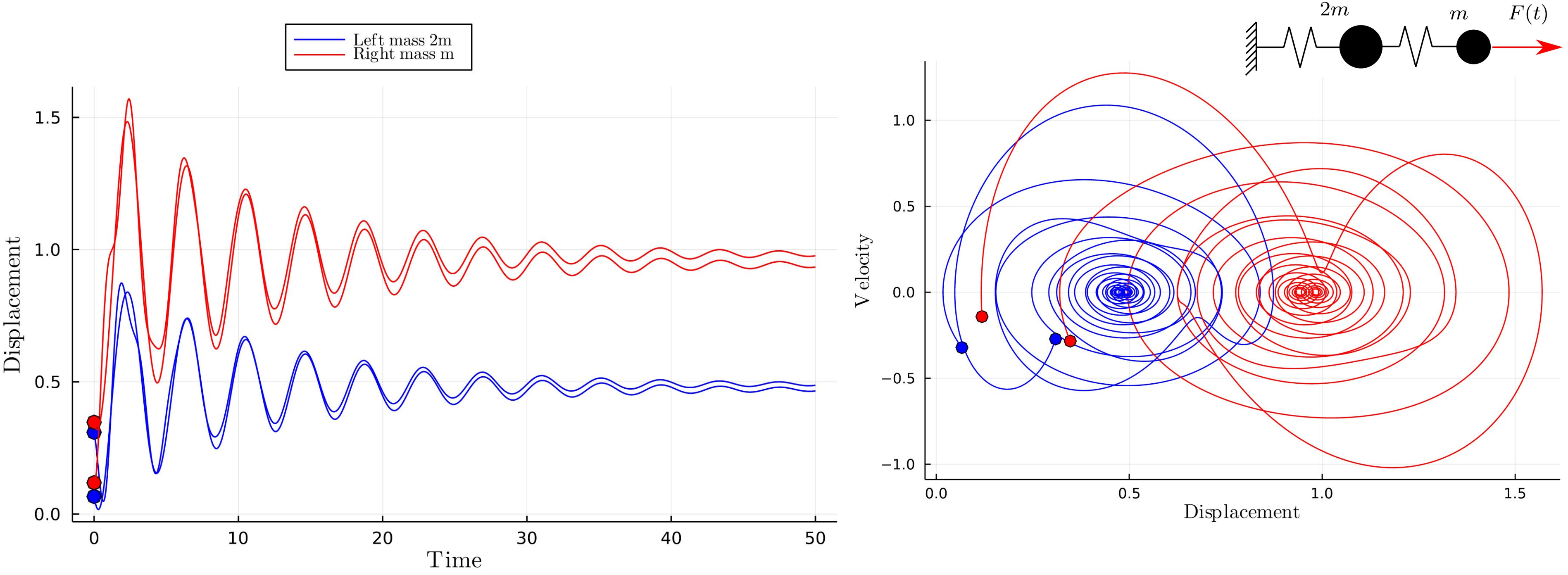
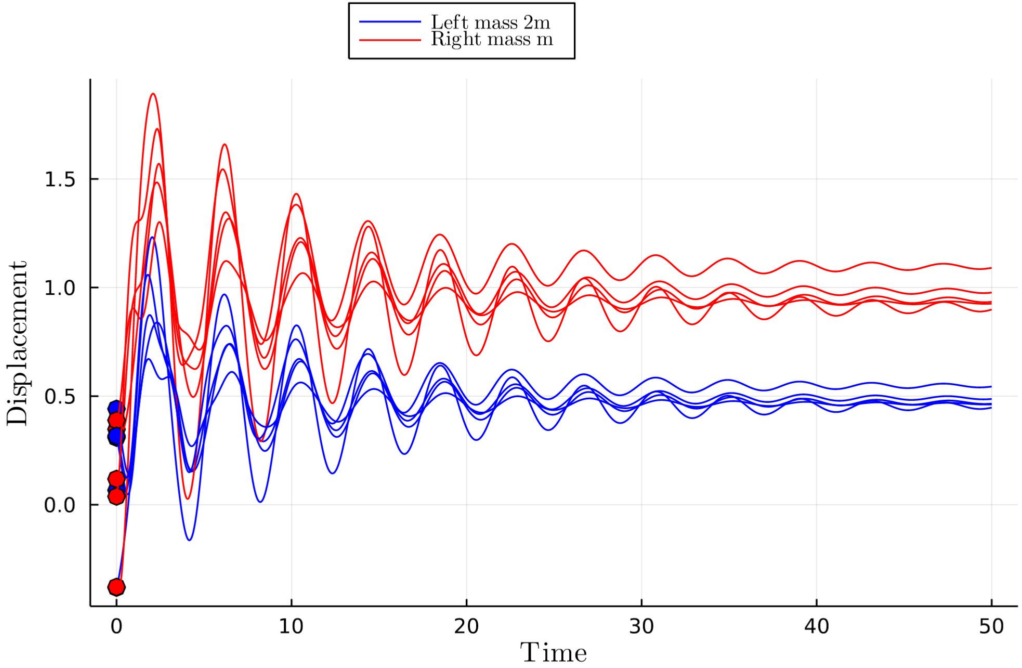
Pequeña perturbación de las CI

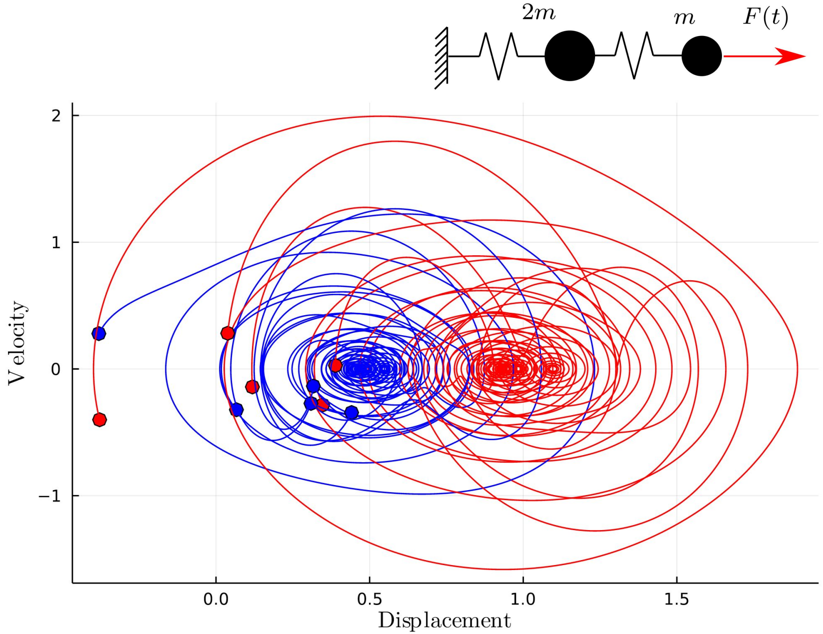


Github: https://github.com/JuliaReach/SetPropagation-FEM-JuliaCon21

Forets, M., & Caporale, D. F. (2022, June). Computing Reachable Sets of Semi-Discrete Solid Dynamics Equations with ReachabilityAnalysis.jl. In Proceedings of the JuliaCon Conferences (Vol. 1, No. 1, p. 95).

Mayor perturbación de las CI





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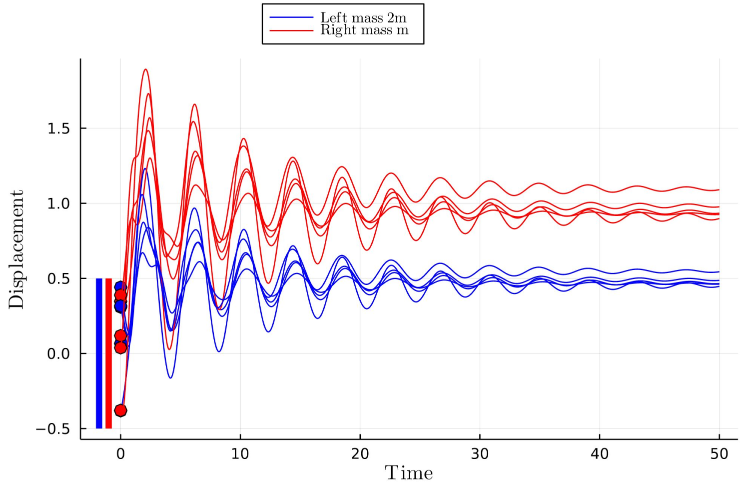
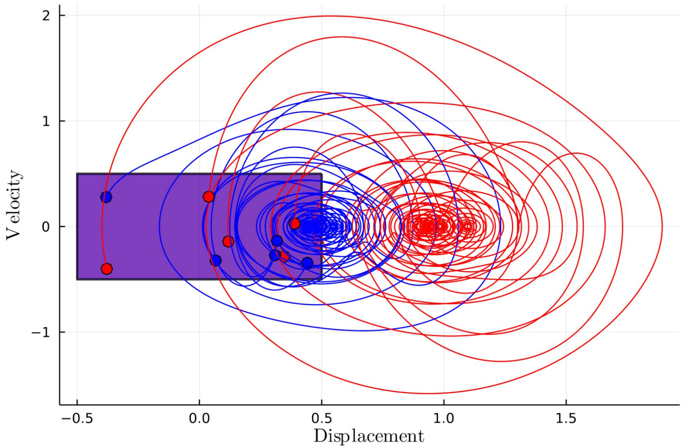
¿Cómo capturar todos los comportamientos admisibles?
Propagar **conjuntos** en vez de **puntos** >> Reachability analysis





Reachability is a numerical method to reason about **uncertainty** in the sets of states reachable by dynamical systems.

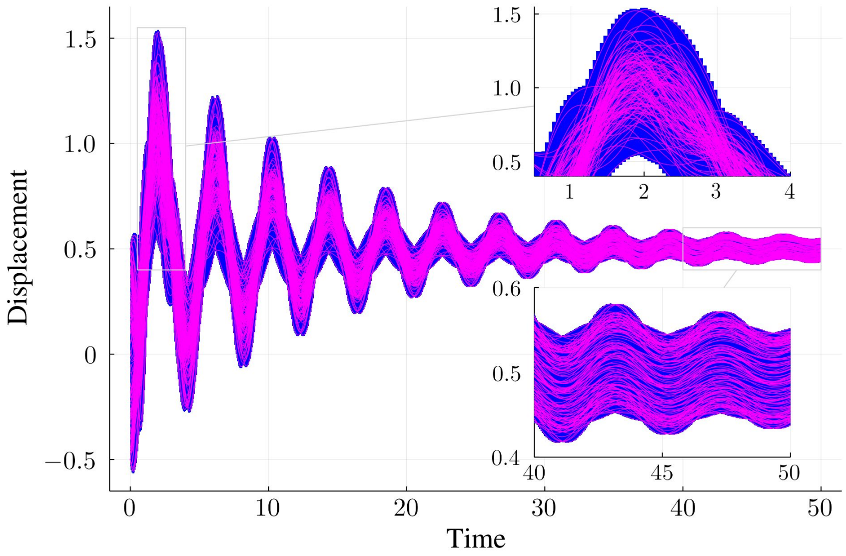
Initial states

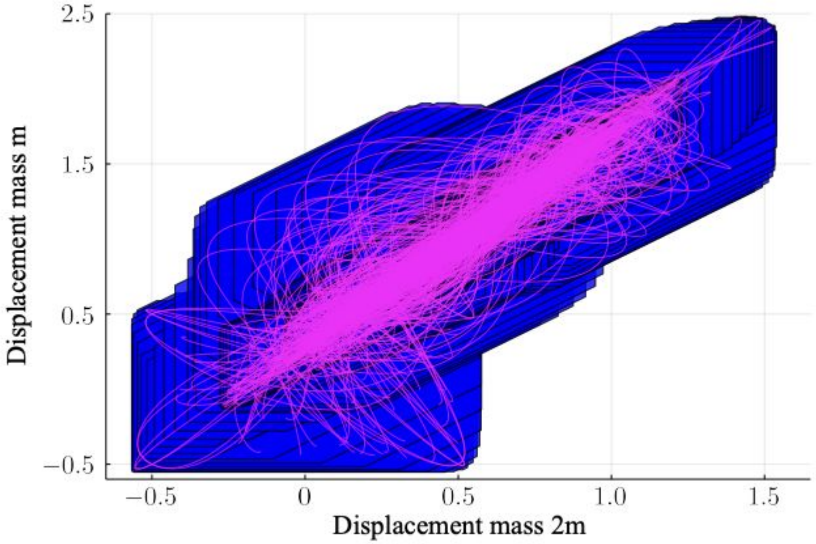
Admissible parameters

Admissible inputs

Ejemplo

$$\text{Reach}_{[0,T]}(\mathcal{X}_0) = \bigcup_{t \in [0,T]} \text{Reach}_t(\mathcal{X}_0)$$

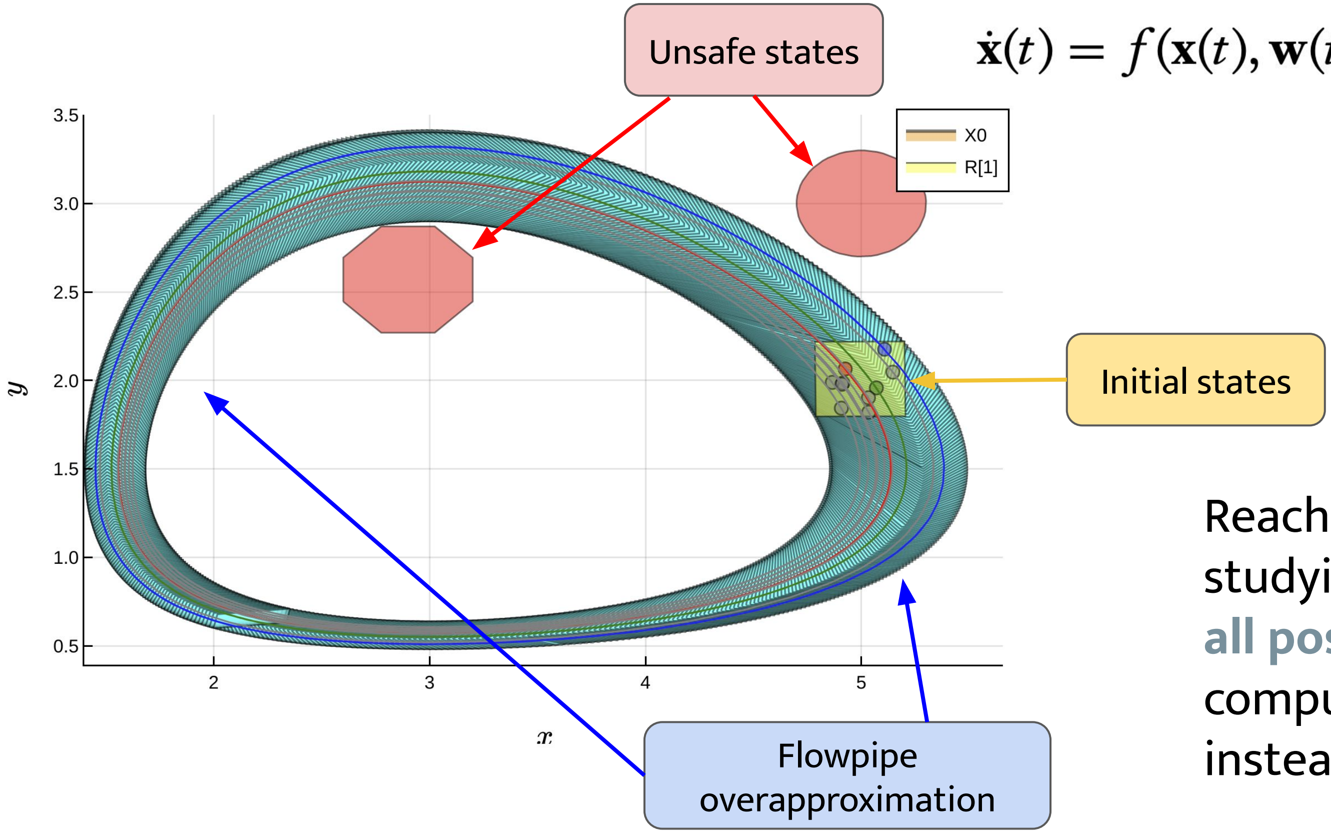




J. Perez Zerpa, M. Forets, and D. F. Caporale. "Computing Reachable Sets of Semi-Discrete Solid Dynamics Equations with ReachabilityAnalysis. jl." In *Proceedings of the JuliaCon Conferences*, vol. 1, no. 1, p. 95. 2022.

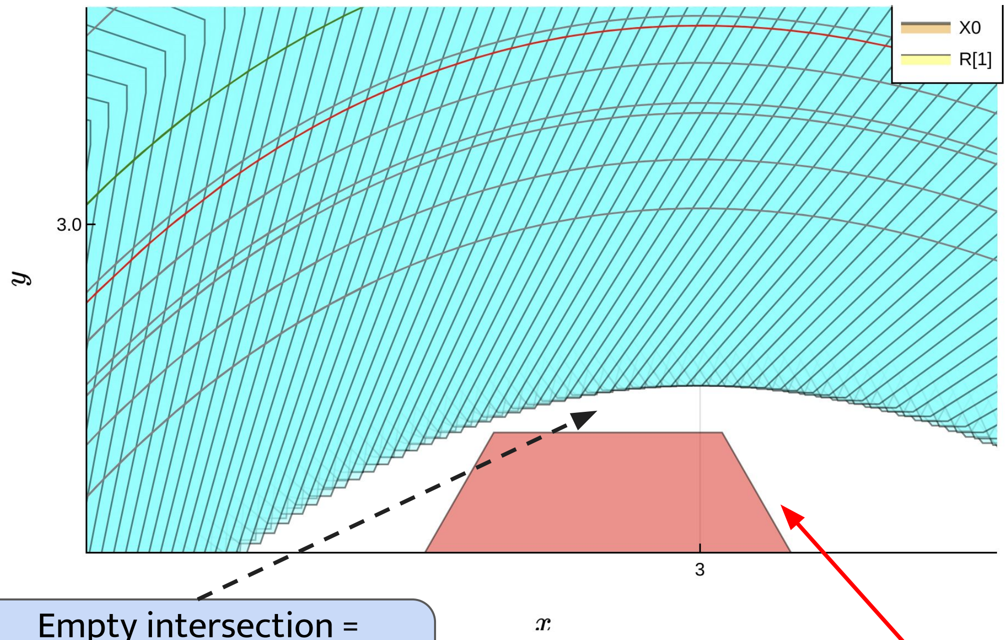
¿Para qué sirve? Caso de uso: verificación algorítmica

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{w}(t)),\ \mathbf{x}(t) \in \mathbb{R}^n,\ \mathbf{w}(t) \in \mathcal{W}$$



Reachability analysis: studying the set of **all possible behaviors**, computing with **sets** instead of scalars

¿Para qué sirve? Caso de uso: verificación algorítmica



Empty intersection = **algorithmic proof** that the system is safe

Unsafe states

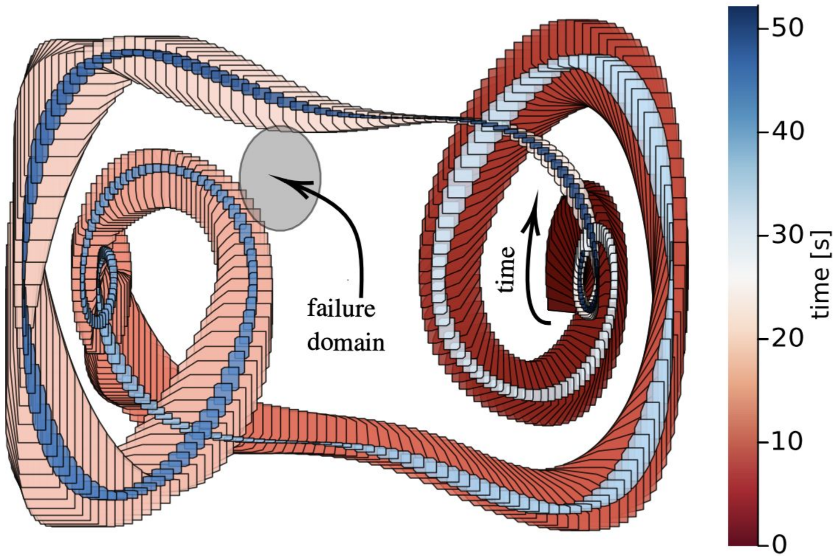
$$x'(t) = f(x(t), u(t), p(t))$$

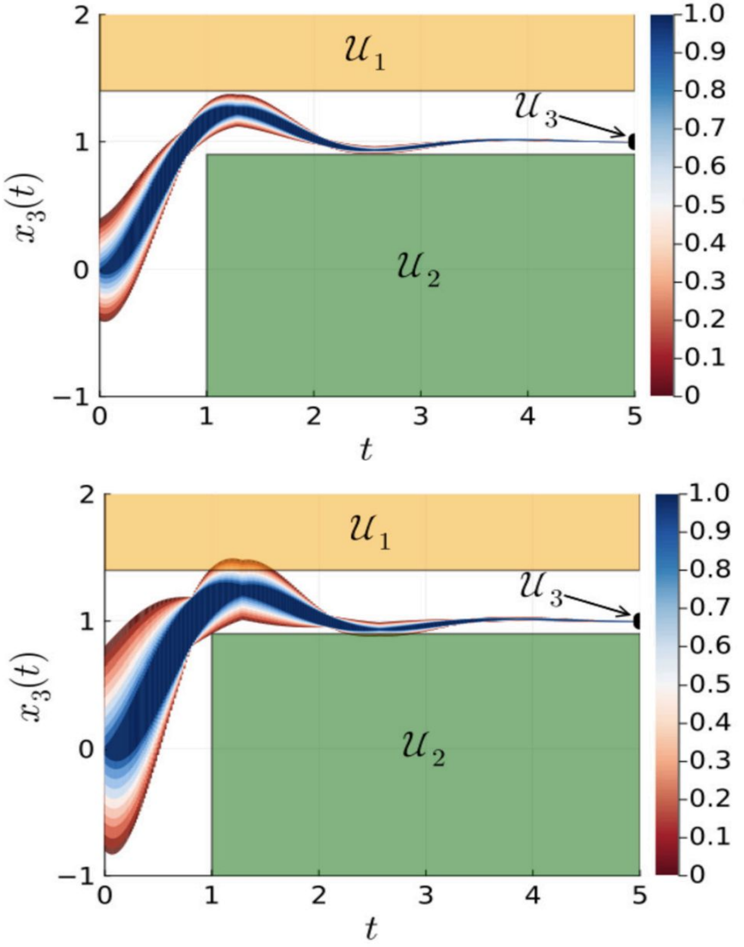
Is there a trajectory such that the **solution enters the unsafe set** within the given time bound?

¿Para qué sirve?

- **Formal verification.**

- **Controller synthesis.**

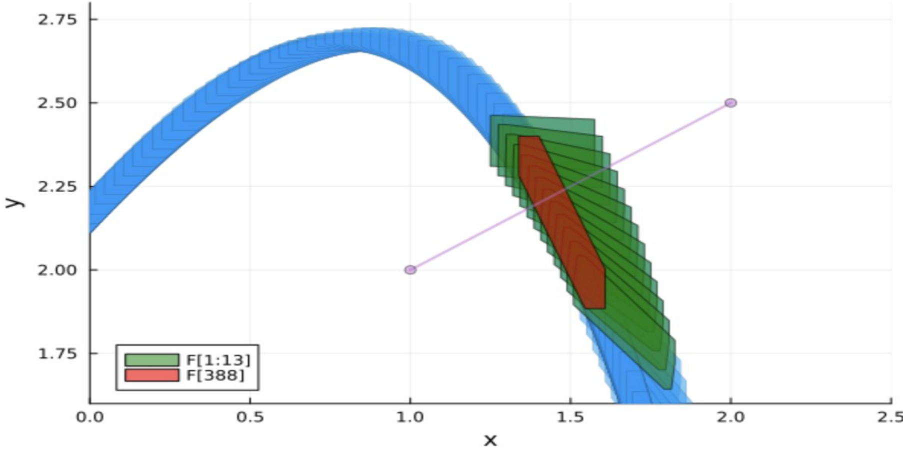


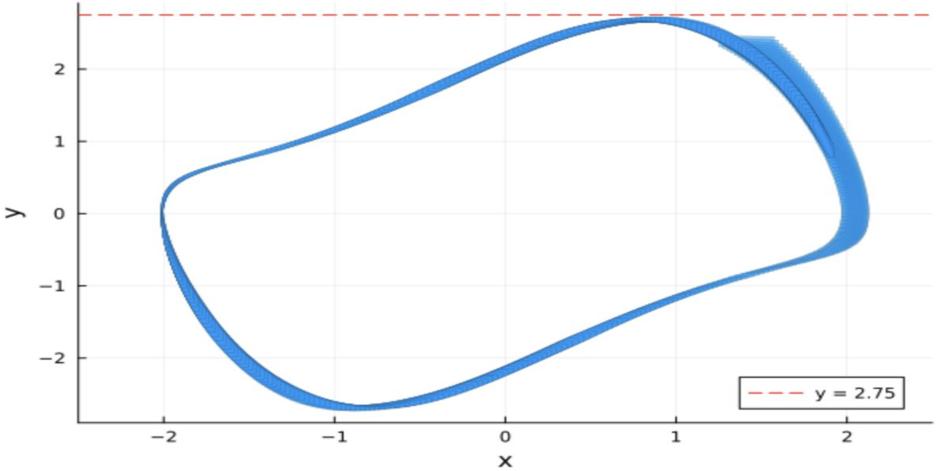


Gray, Ander, Marcelo Forets, Christian Schilling, Scott Ferson, and Luis Benet. "Verified propagation of imprecise probabilities in non-linear ODEs." *International Journal of Approximate Reasoning* 164 (2024): 109044.

¿Para qué sirve?

- **Formal verification.**
- **Controller synthesis.**
- **Computation of invariant sets and regions of attraction.**





Marcelo Forets, Christian Schillin, Luis Benet et al. ReachabilityAnalysis.jl (2024)

¿Para qué sirve?

- **Formal verification.**
- **Controller synthesis.**
- **Computation of invariant sets and regions of attraction.**
- **Robust control.**

Provably safe human-robot interaction

Althoff, Matthias, Goran Frehse, and Antoine Girard. "Set propagation techniques for reachability analysis." Annual Review of Control, Robotics, and Autonomous Systems 4, no. 1 (2021): 369-395.

¿Para qué sirve?

- **Formal verification.**
- **Controller synthesis.**
- **Computation of invariant sets and regions of attraction.**
- **Robust control.**
- **Fault detection.**
- **Conformance checking.**
- . . .

Guaranteeing safety in autonomous systems.

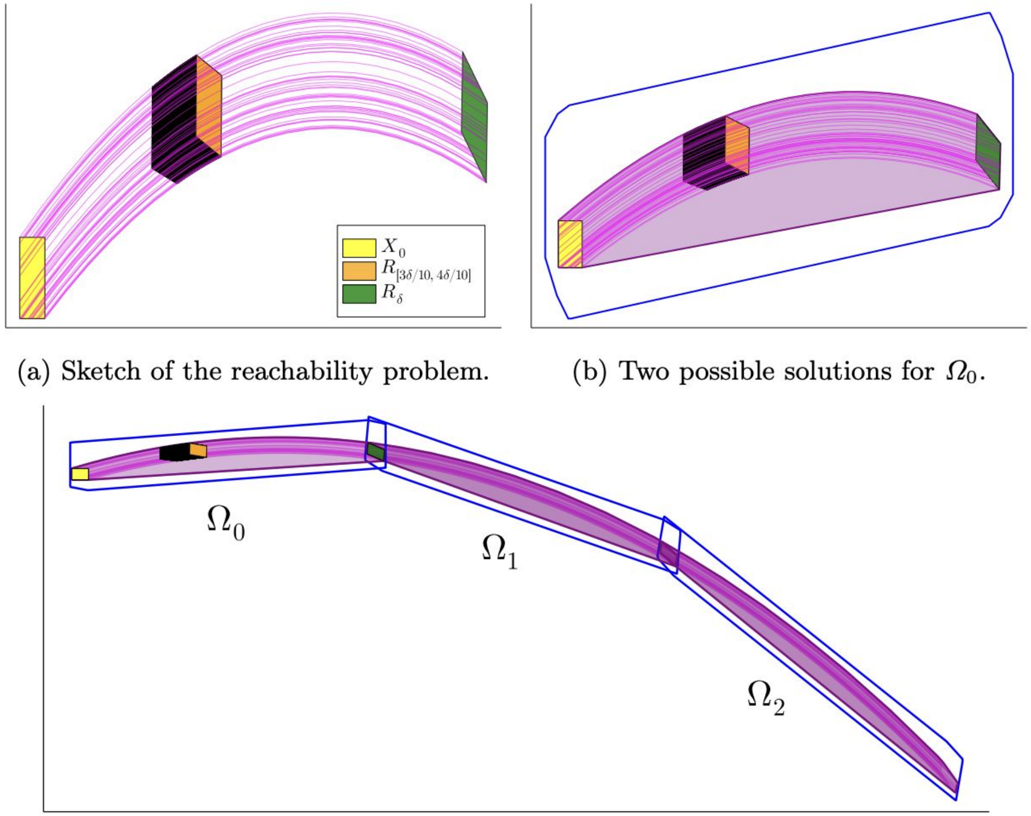
Althoff, Matthias, Goran Frehse, and Antoine Girard. "Set propagation techniques for reachability analysis." Annual Review of Control, Robotics, and Autonomous Systems 4, no. 1 (2021): 369-395.

Idea de Set Propagation

- Reducir la ODE cont. a un sistema de ecuaciones en recurrencias entre conjuntos de R^n.

- Resolviendo dicho sistema obtengo una sobre-aproximación del flowpipe exacto representada por la unión de las soluciones.

- Se demuestra convergencia empleando la distancia de Hausdorff.



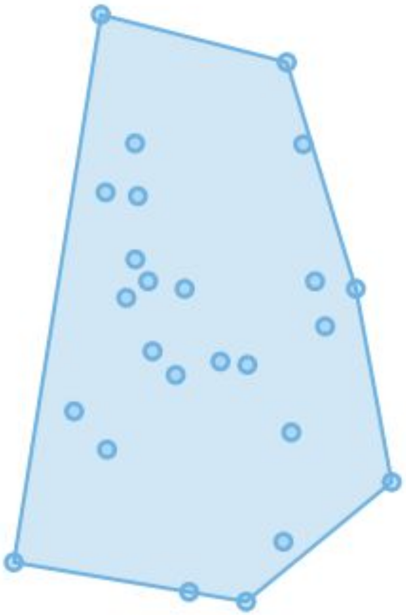
(a) Sketch of the reachability problem.

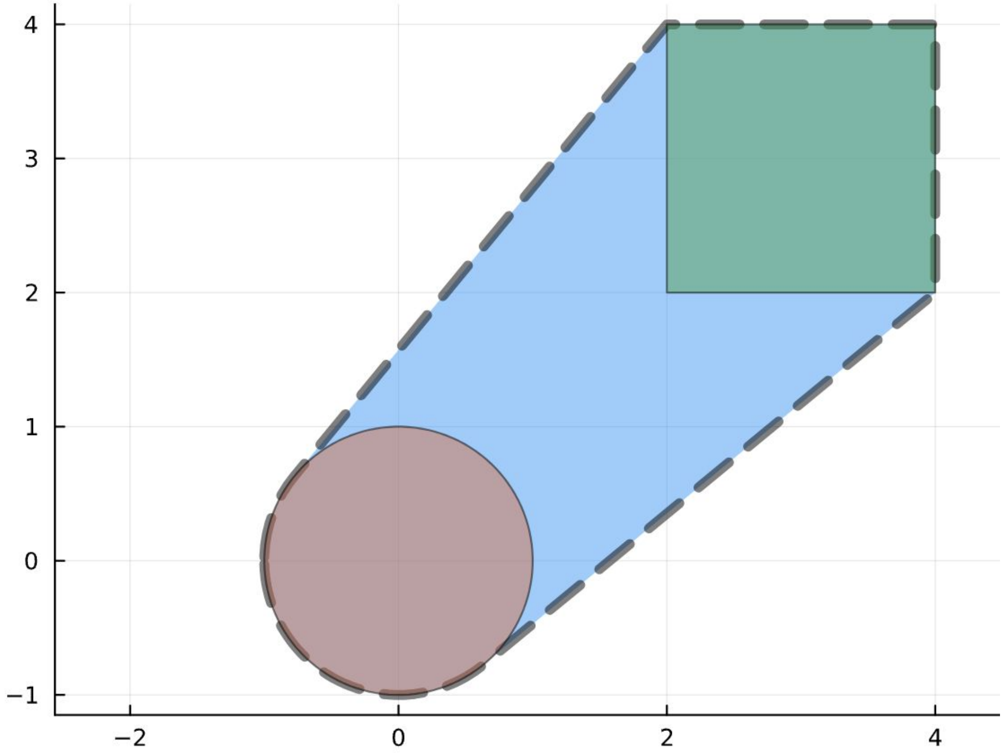
(b) Two possible solutions for Ω_0.

(c) First three steps of the iteration in a reachability algorithm.

Forets, M. and Schilling, C., 2022, June. Conservative time discretization: a comparative study. In International Conference on Integrated Formal Methods (pp. 149-167). Cham: Springer International Publishing.

Operaciones básicas

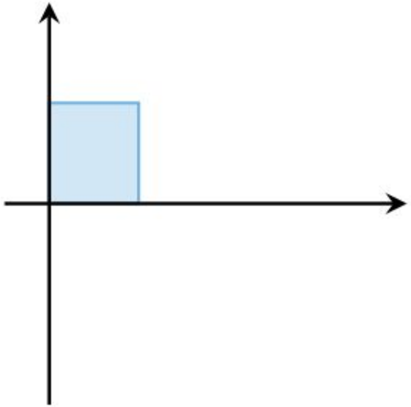




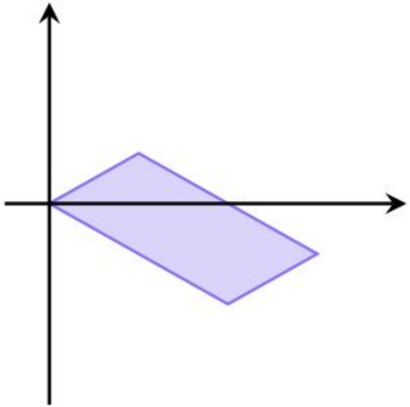
Envolvente convexa (Convex Hull, CH)

Operaciones básicas

$$\begin{bmatrix} 1.0 & 2.0 \\ 0.5 & -1.0 \end{bmatrix} \times$$

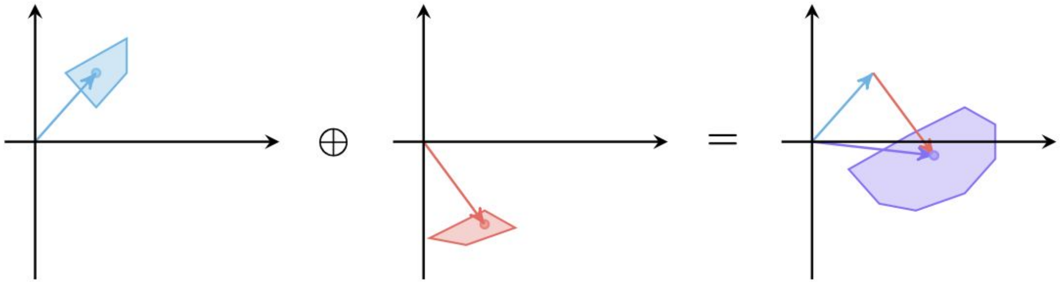


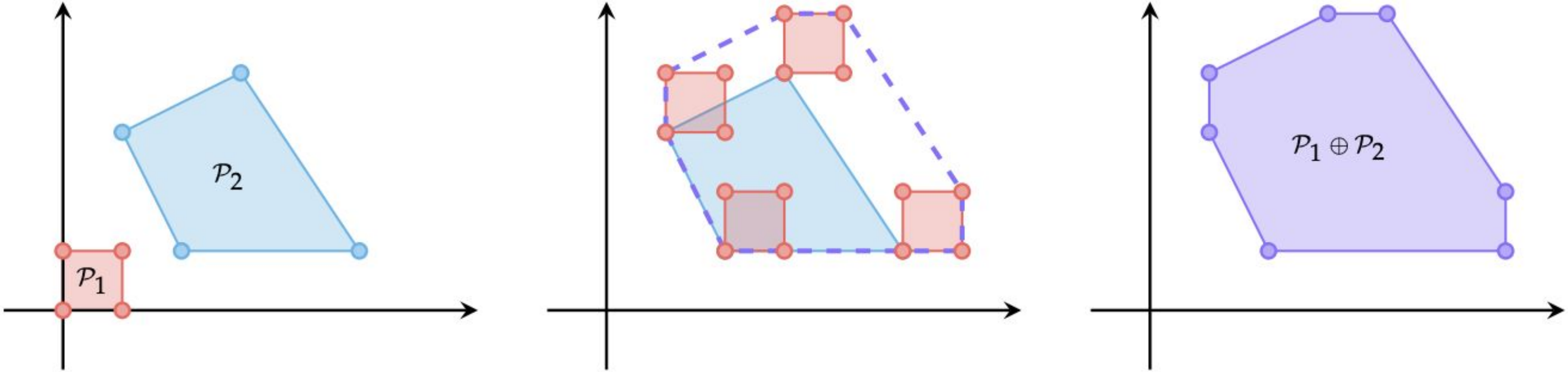
$$=$$



Transformación lineal

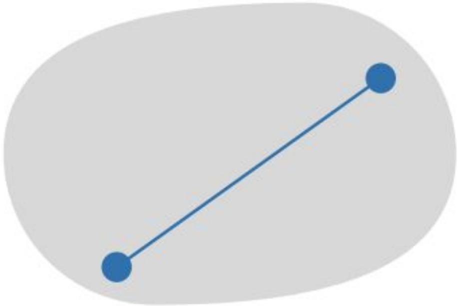
Operaciones básicas



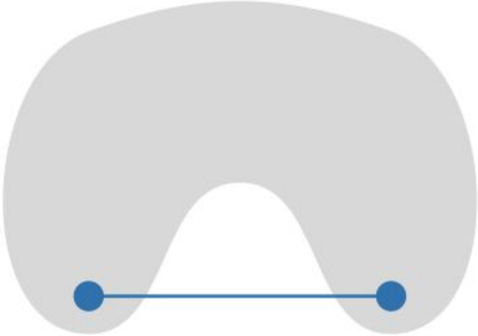


Suma de Minkowski

Representaciones

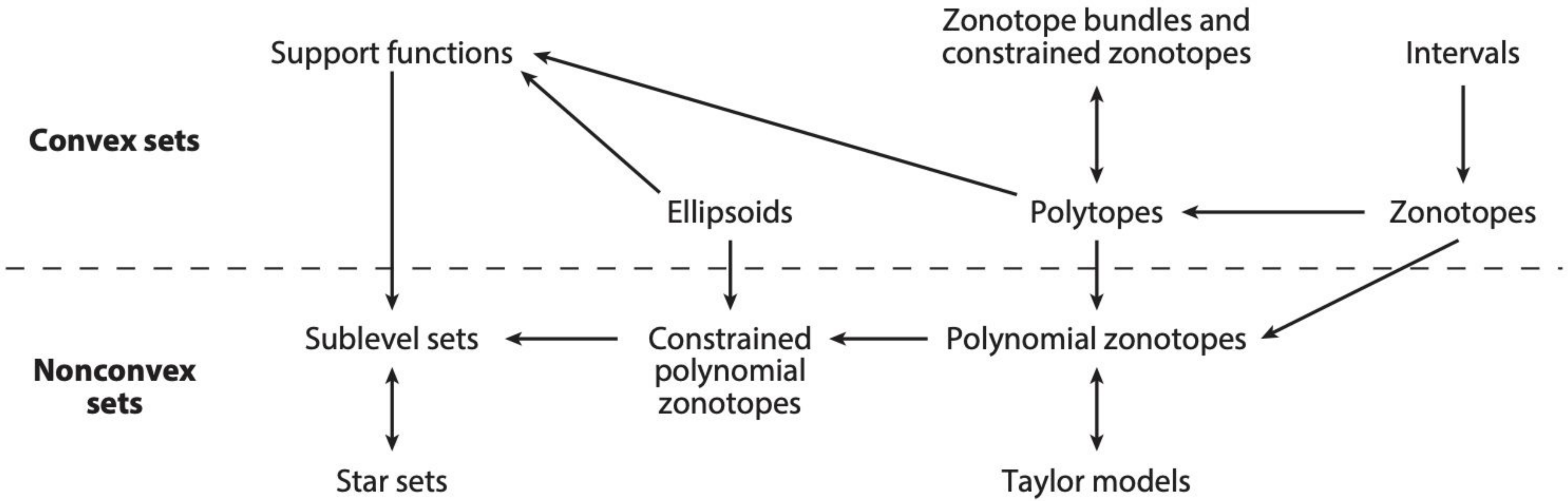


a convex set



a nonconvex set

Zoo de Representaciones



Althoff, Matthias, Goran Frehse, and Antoine Girard. "Set propagation techniques for reachability analysis." Annual Review of Control, Robotics, and Autonomous Systems 4, no. 1 (2021): 369-395.

Zoo de operaciones

Operation name	Math form	Lazy function (constructor)	Short form	Concrete function
Minkowski sum[b]	$X \oplus Y$	`MinkowskiSum`	`+`, `\oplus`	`minkowski_sum`
Intersection[b]	$X \cap Y$	`Intersection`	`\cap`	`intersection`
Cartesian product[b]	$X \times Y$	`CartesianProduct`	`*`, `\times`	`cartesian_product`
Convex hull[b]	$CH(X \cup Y)$	`ConvexHull`	`CH`	`convex_hull`
Symmetric interval hull	$\boxdot(X)$	`SymmetricIntervalHull`	`\boxdot`	`symmetric_interval_hull`
Linear map	AX	`LinearMap`	`*`	`linear_map`
Exponential map	$e^A X$	`ExponentialMap`		`exponential_map`
Translation	$X + b$	`Translation`	`+`	`translate`
Affine map	$AX + b$	`AffineMap`	`*` and `+`	`affine_map`
Reset map	$x_i \mapsto c$	`ResetMap`		-
Inverse linear map	$A^{-1}X$	`InverseLinearMap`		-
Bloating	$X \oplus \{x : \|x\| \leq \varepsilon\}$	`Bloating`		-
Union[b]	$X \cup Y$	`UnionSet`	`\cup`	-
Complement	X^C	`Complement`		`complement`
Rectified linear unit	$x_i \mapsto \max(x_i, 0)$	`Rectification`		`rectify`

Forets, Marcelo, and Christian Schilling. "LazySets.jl: Scalable Symbolic-Numeric Set Computations." *Proceedings of JuliaCon* 1 (2021): 1.

JuliaReach

The **JuliaReach** open-source project

JuliaReach

https://github.com/JuliaReach/

To advance state-of-the-art working on *fundamental* problems

To build comprehensive, efficient, correct, reproducible, well documented libraries

To widen the applicability of reachability analysis for scientists & engineers

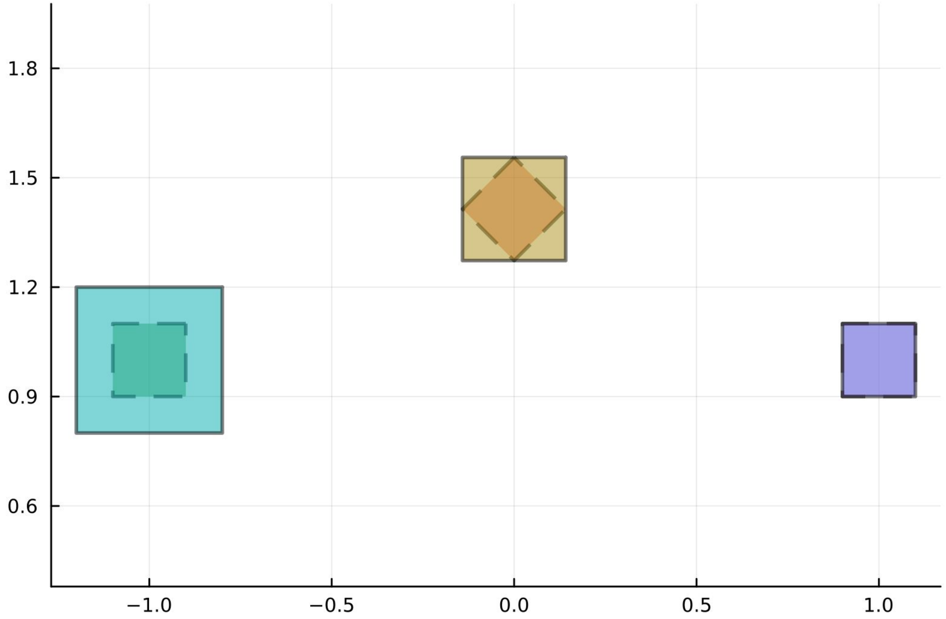
Intervals and Hyperrectangles (convex)

$$H = \{\mathbf{x} \in \mathbb{R}^n : |x_i - c_i| \leqslant r_i, \quad i = 1, \ldots, n\}$$

Eficiente +

Poca expresividad −

Efecto de wrapping −



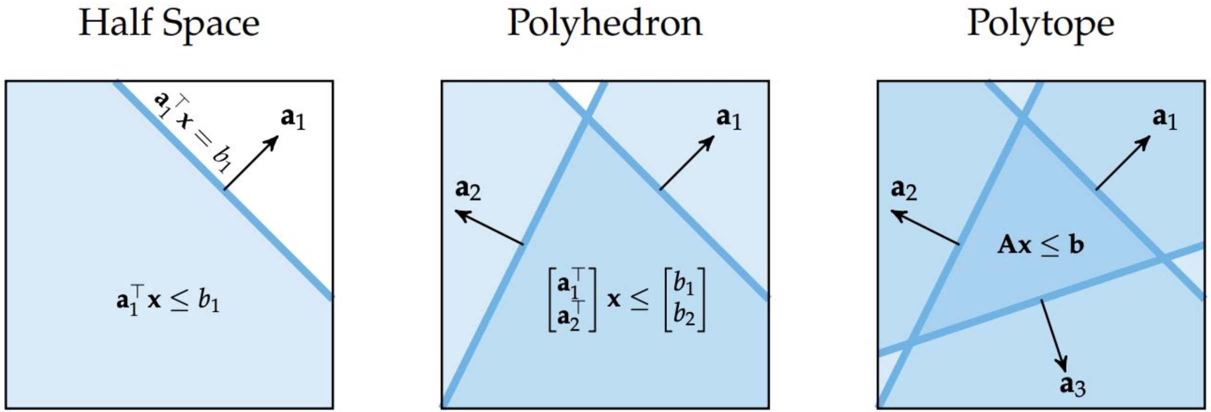
Polytopes and polyhedra (convex)

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n \,\middle|\, Ax \leq b \right\}$$

Dualidad H-rep y V-rep:

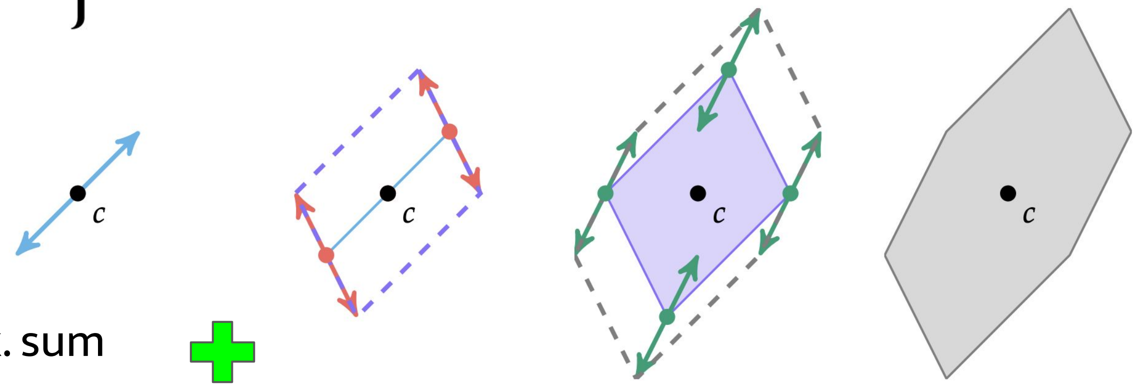
Interseccion en H-rep

Linear map en V-rep



Zonotopes (convex)

$$\mathcal{Z} := \left\{ c + \sum_{i=1}^{p} \alpha_i G_{(\cdot,i)} \,\middle|\, \alpha_i \in [-1, 1] \right\}$$



Cerrada bajo linear map y mink. sum

Admite rep. comp. eficiente

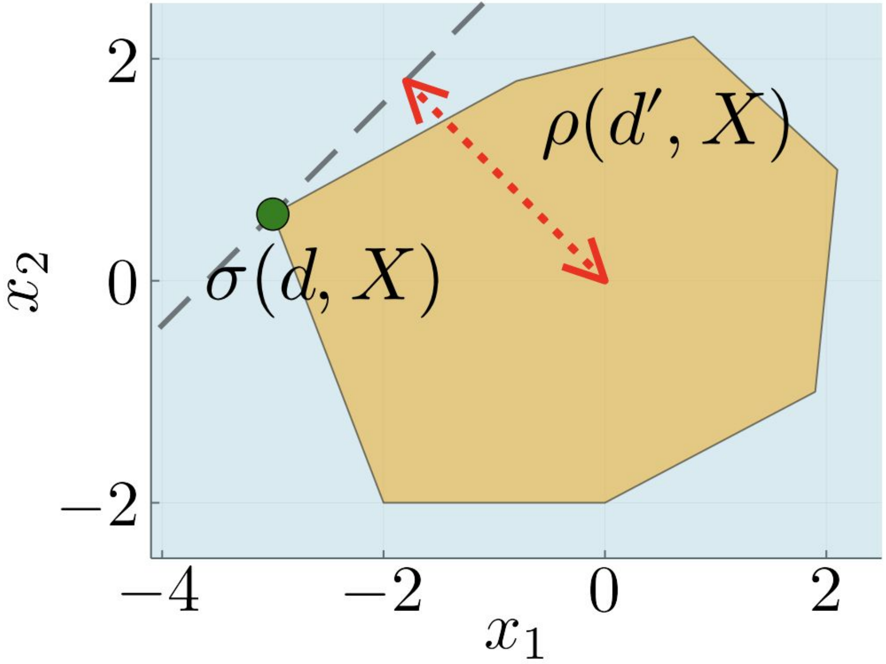
No es cerrada bajo intersección

Costo computacional

Class	Linear map	Minkowski sum
Intervals	x	$\mathcal{O}(n)$ (43, equation 2.67)
Ellipsoids	$\mathcal{O}(\max(mn^2, m^2n))$ (44, section 2.2.1)	x
$\mathcal{H}$-polytopes (spanned by n generators)	$\mathcal{O}(n^3)$[a] (45, table 1)	$\mathcal{O}(2^n)$ (45, table 1)
$\mathcal{V}$-polytopes (spanned by n generators)	$\mathcal{O}(mn2^n)$ (45, table 1)	$\mathcal{O}(n2^{2n})$ (45, table 1)
Zonotopes (n generators)	$\mathcal{O}(mn^2)$ (49, table I)	$\mathcal{O}(n)$ (49, table I)
Z[illegible]	$\mathcal{O}(n^3)$[a] (38[illegible]	

Support functions (convex, lazy)

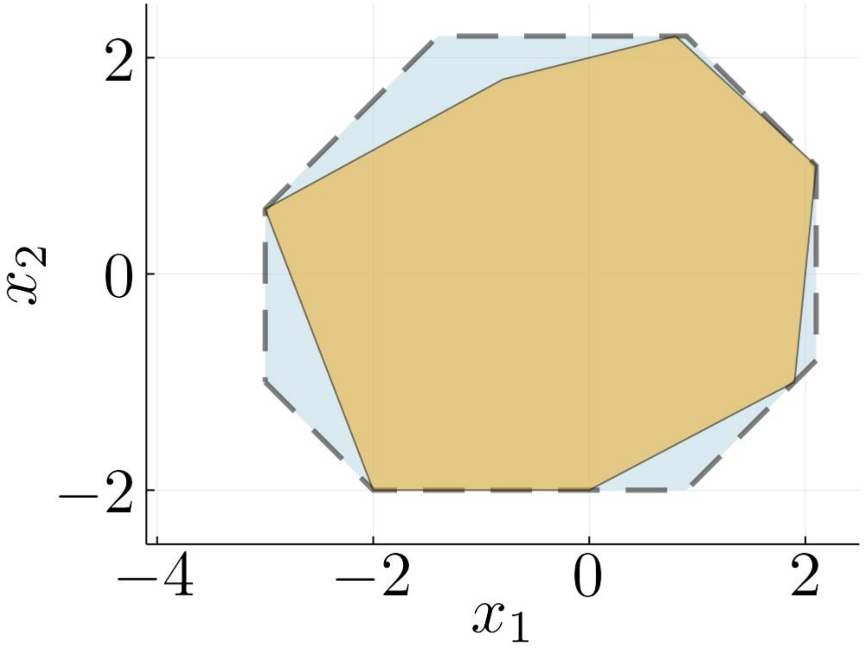
$$\rho_{\mathcal{S}}(\ell) = \max_{x \in \mathcal{S}} \ell^\top x.$$



Support functions (convex, lazy)

$$\rho_{\mathcal{S}}(\ell) = \max_{x \in \mathcal{S}} \ell^\top x.$$

$$\mathcal{S} \subseteq \bigcap_{\ell \in \mathcal{L}} \left\{ x \in \mathbb{R}^n \middle| \ \ell^\top x \leq \rho_{\mathcal{S}}(\ell) \right\}$$



Support functions (convex, lazy)

$$\rho_{\mathcal{S}}(\ell) = \max_{x \in \mathcal{S}} \ell^\top x.$$

$$\rho_{\lambda\mathcal{X}}(\ell) = \rho_{\mathcal{X}}(\lambda\ell), \text{ and } \rho_{\lambda\mathcal{X}}(\ell) = \lambda\rho_{\mathcal{X}}(\ell) \text{ if } \lambda > 0$$

$$\rho_{M\mathcal{X}}(\ell) = \rho_{\mathcal{X}}(M^{\mathrm{T}}\ell)$$

$$\rho_{\mathcal{X}\oplus\mathcal{Y}}(\ell) = \rho_{\mathcal{X}}(\ell) + \rho_{\mathcal{Y}}(\ell)$$

$$\rho_{\mathcal{X}\times\mathcal{Y}}(\ell) = \ell^{\mathrm{T}}\sigma_{\mathcal{X}\times\mathcal{Y}}(\ell) = \rho_{\mathcal{X}}(\ell_1^{\mathrm{T}}) + \rho_{\mathcal{Y}}(\ell_2^{\mathrm{T}})$$

$$\rho_{\mathrm{CH}(\mathcal{X}\cup\mathcal{Y})}(\ell) = \max(\rho_{\mathcal{X}}(\ell), \rho_{\mathcal{Y}}(\ell))$$

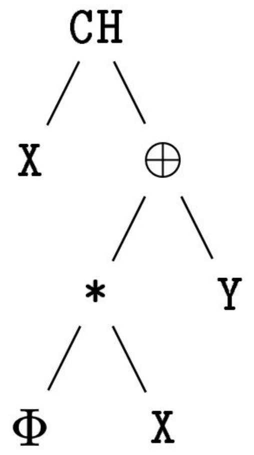
Muy eficiente al ser lazy +

Adaptable (template polyhedra) +

Representación computacional

```
A = [0 1; -(4π)^2 0]
X_0 = BallInf([1.0, 0.0], 0.1)
δ = 0.025
Φ = exp(A*δ)
  2×2 Matrix{Float64}:
    0.95105652  0.02459079
   -3.88322208  0.95105652

r = [0.05477208, 0.07676220]
E_+ = Hyperrectangle(zeros(2), r)
Ω_0 = CH(X_0, Φ*X_0 ⊕ E_+)
```



Reachability para sistemas lineales

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{w}(t),\ \mathbf{x}(t) \in \mathbb{R}^n,\ \mathbf{w}(t) \in \mathcal{W} \subseteq \mathbb{R}^m.$$

Reachability para sistemas lineales

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{w}(t),\ \mathbf{x}(t) \in \mathbb{R}^n,\ \mathbf{w}(t) \in \mathcal{W} \subseteq \mathbb{R}^m$$

$$\mathsf{Reach}_{[0,T]}(\mathcal{X}_0) = \bigcup_{t\in[0,T]} \mathsf{Reach}_t(\mathcal{X}_0)$$

$$N \in \mathbb{N}$$

$$\tau = T/N$$

$$\boxed{\mathsf{Reach}_{[0,T]}(\mathcal{X}_0) = \bigcup_{k=0}^{N-1} \mathsf{Reach}_{k\tau}(\mathsf{Reach}_{[0,\tau]}(\mathcal{X}_0))}$$

Reachability para sistemas lineales

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{w}(t),\ \mathbf{x}(t) \in \mathbb{R}^n,\ \mathbf{w}(t) \in \mathcal{W} \subseteq \mathbb{R}^m.$$

$$\mathsf{Reach}_{[0,T]}(\mathcal{X}_0) = \bigcup_{t \in [0,T]} \mathsf{Reach}_t(\mathcal{X}_0)$$

$N \in \mathbb{N}$

$\tau = T/N$

$$\boxed{\mathsf{Reach}_{[0,T]}(\mathcal{X}_0) = \bigcup_{k=0}^{N-1} \mathsf{Reach}_{k\tau}(\mathsf{Reach}_{[0,\tau]}(\mathcal{X}_0)).}$$

Idea: encontrar $(\mathcal{S}_k)_{k=0}^{N-1}$ tal que Sk contiene a $\boxed{\mathsf{Reach}_{k\tau}(\mathsf{Reach}_{[0,\tau]}(\mathcal{X}_0)).}$

Reachability para sistemas lineales

Idea: encontrar $(\mathcal{S}_k)_{k=0}^{N-1}$ tal que Sk contiene a $\boxed{\mathsf{Reach}_{k\tau}(\mathsf{Reach}_{[0,\tau]}(\mathcal{X}_0)).}$

Solución: existen varios esquemas de
Discretización temporal conservativa [1]

$$\boxed{\mathcal{S}_0 = \operatorname{conv}\left(\mathcal{X}_0 \cup (e^{A\tau}\mathcal{X}_0 \oplus \tau\mathcal{W} \oplus \alpha(\tau, A, B, \mathcal{X}_0, \mathcal{W})\mathcal{B})\right),}$$

[1] Forets, Marcelo, and Christian Schilling. "Conservative time discretization: a comparative study." In *International Conference on Integrated Formal Methods*, pp. 149-167. Cham: Springer International Publishing, 2022.

Reachability para sistemas lineales

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Reachability para sistemas lineales

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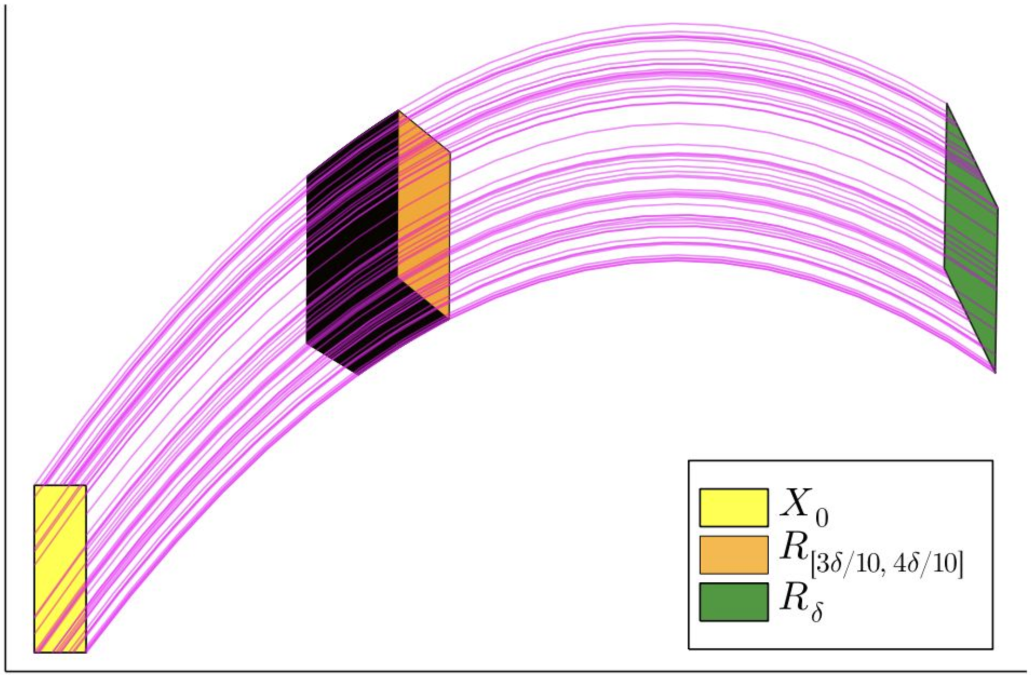
$$\mathcal{S}_k = e^{A\tau}\mathcal{S}_{k-1} \oplus \tau\mathcal{W} \oplus \beta(\tau, A, B, \mathcal{W})\mathcal{B},$$

Resultado:

$$\boxed{\mathsf{Reach}_{[0,T]}(\mathcal{X}_0) \subseteq \mathcal{R}_N \text{ and } d_{\mathrm{H}}(\mathsf{Reach}_{[0,T]}(\mathcal{X}_0), \mathcal{R}_N) = \mathcal{O}(\tau), \text{ where } \mathcal{R}_N = \bigcup_{k=0}^{N-1} \mathcal{S}_k.}$$

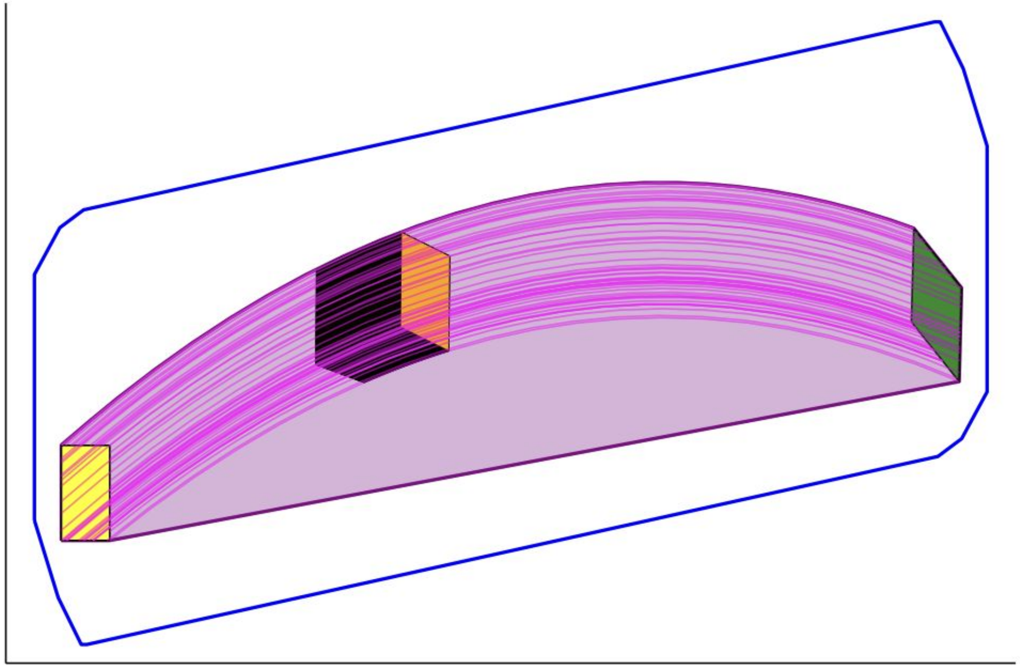
Resumen



(a) Sketch of the reachability problem.

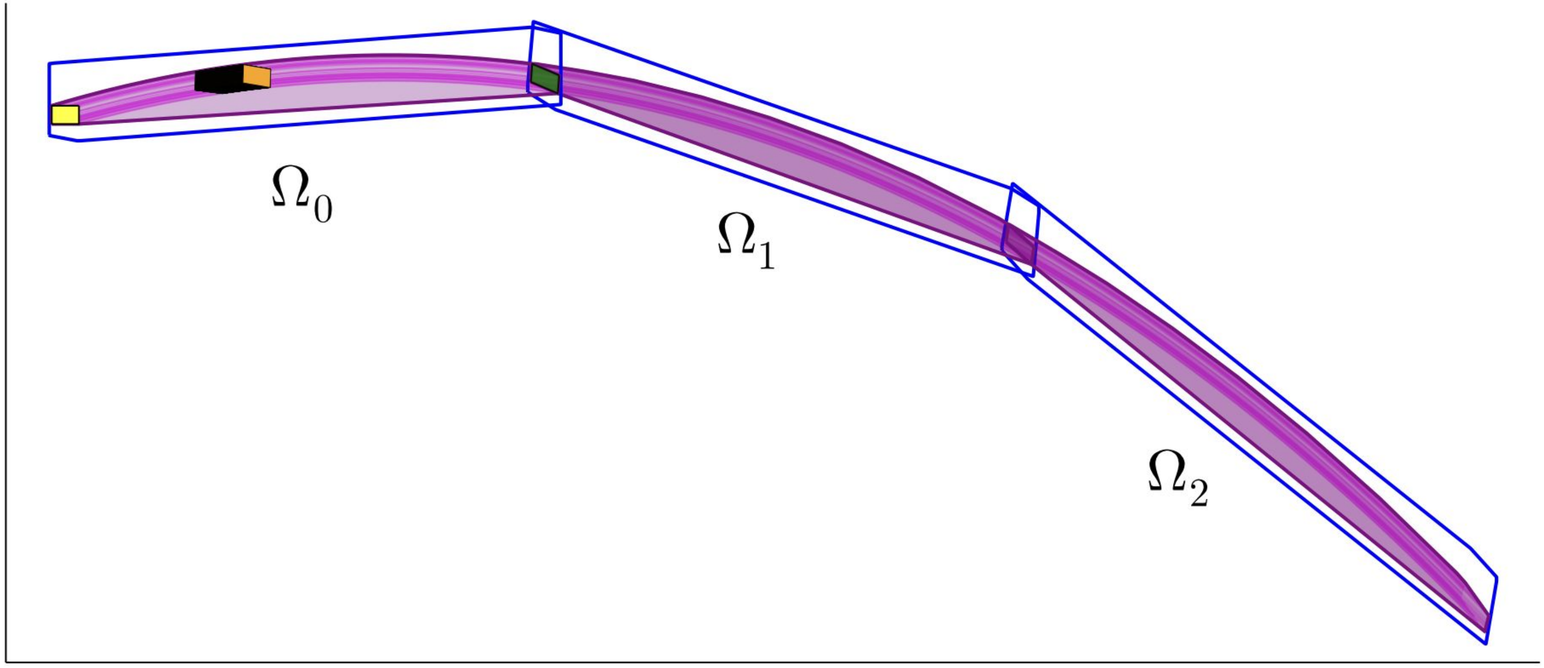
Forets, Marcelo, and Christian Schilling. "Conservative time discretization: a comparative study." In *International Conference on Integrated Formal Methods*, pp. 149-167. Cham: Springer International Publishing, 2022.

Reachability for linear systems = Conservative Discretization + Set propagagtion



(b) Two possible solutions for Ω_0.

Propagación de conjuntos (set propagation)



(c) First three steps of the iteration in a reachability algorithm.

Sin embargo ...

$$\mathcal{S}_0 = \operatorname{conv}\left(\mathcal{X}_0 \cup (e^{A\tau}\mathcal{X}_0 \oplus \tau\mathcal{W} \oplus \alpha(\tau, A, B, \mathcal{X}_0, \mathcal{W})\mathcal{B})\right),$$

$$\mathcal{S}_k = e^{A\tau}\mathcal{S}_{k-1} \oplus \tau\mathcal{W} \oplus \beta(\tau, A, B, \mathcal{W})\mathcal{B},$$

- Dicho esquema tiene un gran problema: la representación se hace cada vez más compleja para cada paso de tiempo!

Sin embargo ...

$$\mathcal{S}_0 = \text{conv}\left(\mathcal{X}_0 \cup (e^{A\tau}\mathcal{X}_0 \oplus \tau\mathcal{W} \oplus \alpha(\tau, A, B, \mathcal{X}_0, \mathcal{W})\mathcal{B})\right)$$

$$\mathcal{S}_k = e^{A\tau}\mathcal{S}_{k-1} \oplus \tau\mathcal{W} \oplus \beta(\tau, A, B, \mathcal{W})\mathcal{B}$$

Dicho esquema tiene un gran problema: la representación se hace cada vez más compleja para cada paso de tiempo!

- Idea: aplicar una reducción en cada paso de tiempo:

$$\boxed{\mathcal{S}_k = \textsf{reduce}\left(e^{A\tau}S_{k-1} \oplus \tau\mathcal{W} \oplus \beta(\tau, A, B, \mathcal{W})\mathcal{B}\right)}$$

Voy de O(N^2) en espacio y tiempo a O(N).

Sin embargo ...

$$\mathcal{S}_k = \textsf{reduce}\left(e^{A\tau} S_{k-1} \oplus \tau \mathcal{W} \oplus \beta(\tau, A, B, \mathcal{W})\mathcal{B}\right)$$

Problema nuevo: efecto de wrapping!

Sin embargo ...

$$\mathcal{S}_k = \textsf{reduce}\left(e^{A\tau} S_{k-1} \oplus \tau\mathcal{W} \oplus \beta(\tau, A, B, \mathcal{W})\mathcal{B}\right)$$

Problema nuevo: efecto de wrapping!

- Si bien el nuevo esquema es O(N), resulta demasiado conservador.

Solución: reorganizar las recurrencias para que la transformación lineal no arrastre el efecto de wrapping.

$$\mathcal{Z}_0 = \text{conv}\left(\mathcal{X}_0 \cup (e^{A\tau}\mathcal{X}_0 \oplus \tau\mathcal{W} \oplus \alpha(\tau, A, B, \mathcal{X}_0, \mathcal{W})\mathcal{B})\right)$$

$$\mathcal{V}_0 = \tau\mathcal{W} \oplus \beta(\tau, A, B, \mathcal{W})\mathcal{B}$$

$$\mathcal{Z}_k = e^{A\tau}\mathcal{Z}_{k-1},\ \mathcal{V}_k = e^{A\tau}\mathcal{V}_{k-1},\ \mathcal{Y}_k = \mathcal{Y}_{k-1} \oplus \mathcal{V}_{k-1} \qquad \mathcal{Y}_0 = \{0\}$$

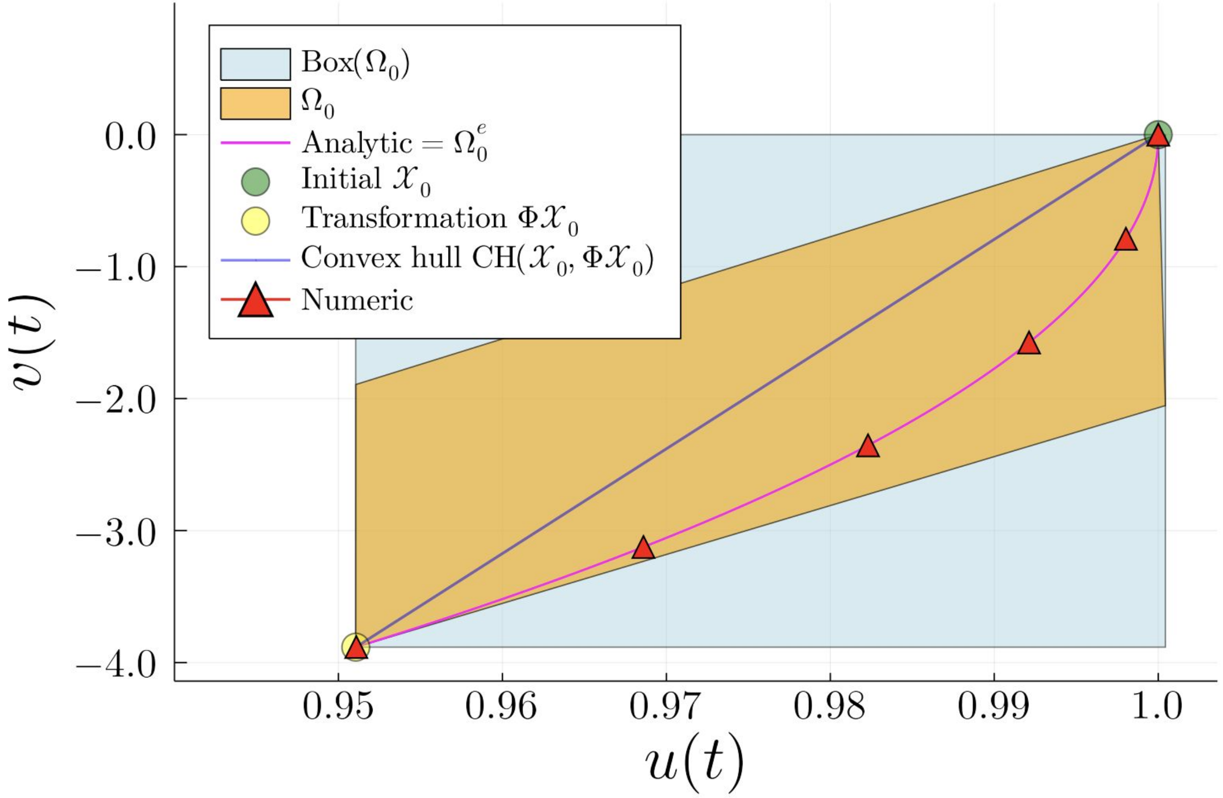
$$\boxed{\mathcal{S}_k = \mathcal{Z}_k \oplus \mathcal{Y}_k} \quad k = 0, \ldots, N-1$$

Ejemplo

$$\ddot{u}(t) + \omega^2 u(t) = 0,$$

$$\mathbf{x}(t) = [u(t), v(t)]^T.$$

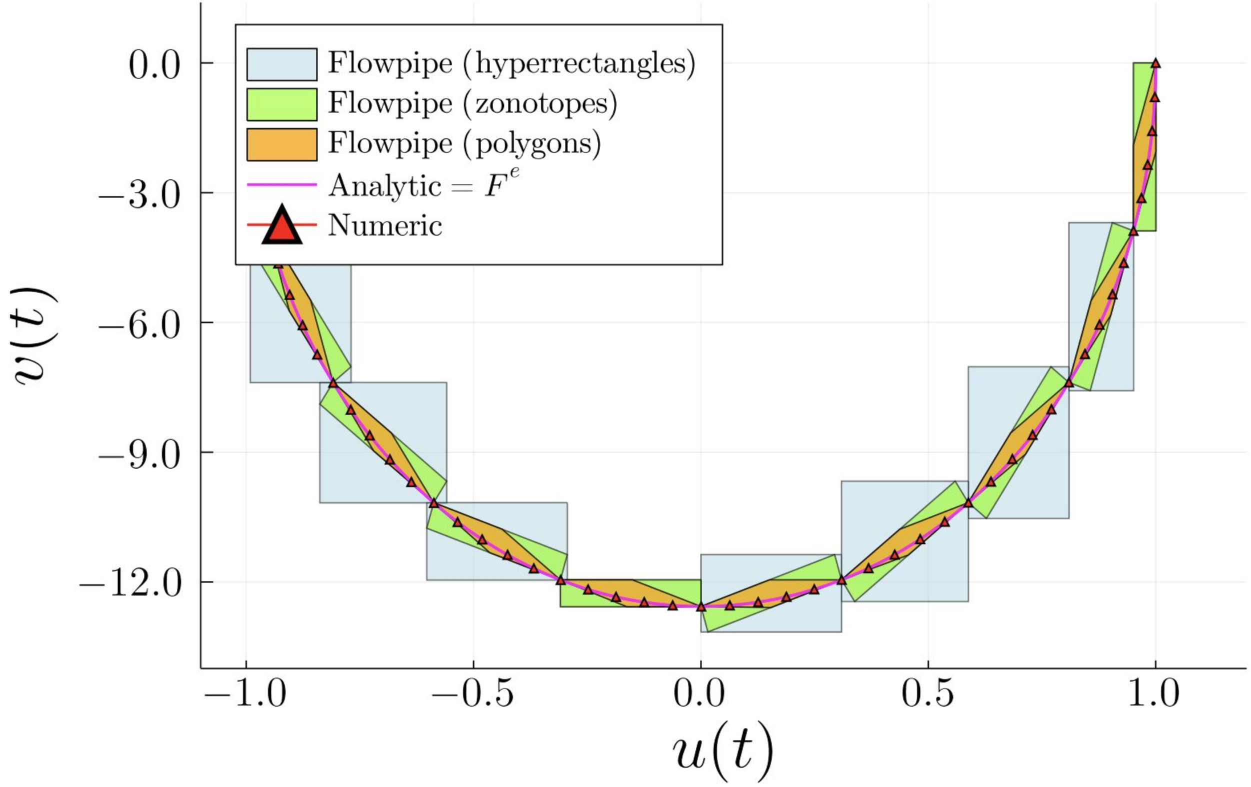
$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \mathbf{x}(t)$$



(a) The set Ω_0 (orange) encloses the true solution (magenta) at the endpoints and at any intermediate time between 0 and $\delta = 0.025$.

Forets, Marcelo, Daniel Freire Caporale, and Jorge M. Pérez Zerpa. "Combining set propagation with finite element methods for time integration in transient solid mechanics problems." *Computers & Structures* 259 (2022): 106699.

Ejemplo



(b) Set propagation using hyperrectangles (lightblue), zonotopes (green) and to assess the zonotopic overapproximation of Ω_0, polygons (orange).

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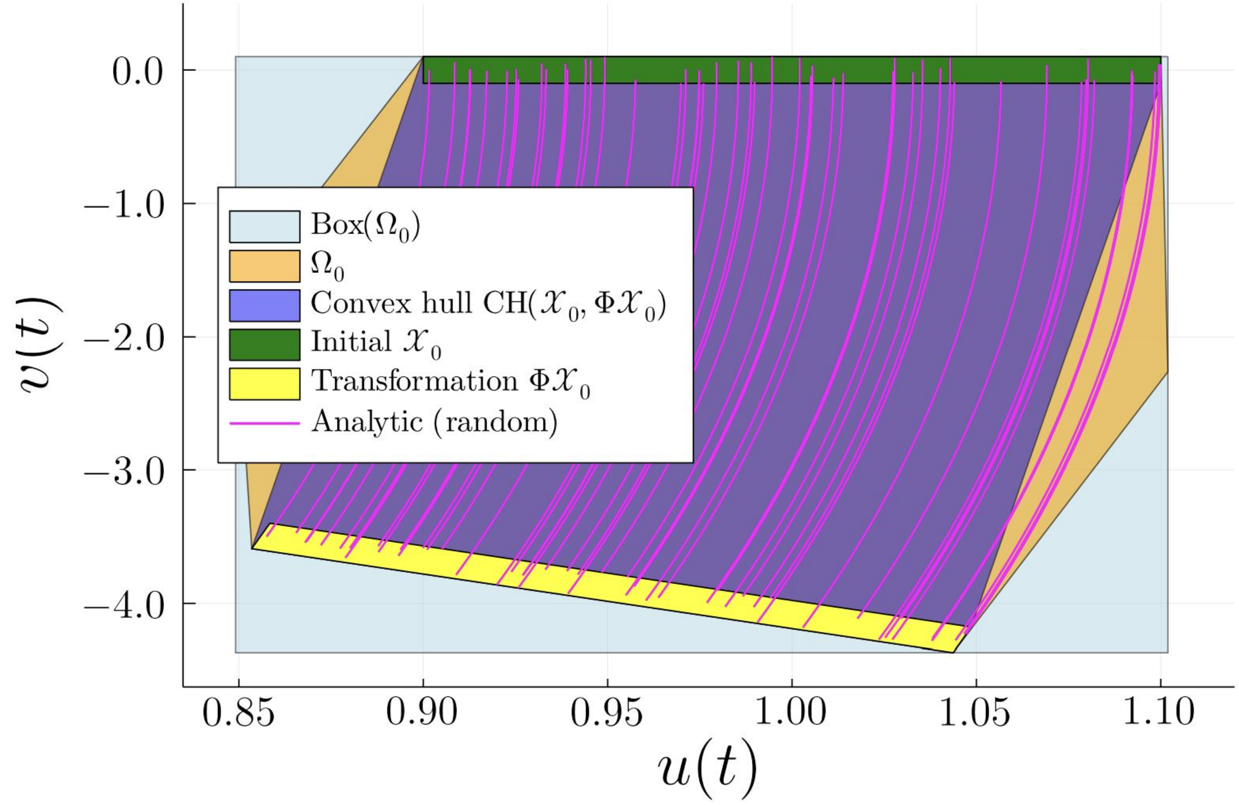
Ejemplo



(a) The set Ω_0 (orange) covers the right-most trajectories for intermediate times, which naturally escape linear interpolations (violet).

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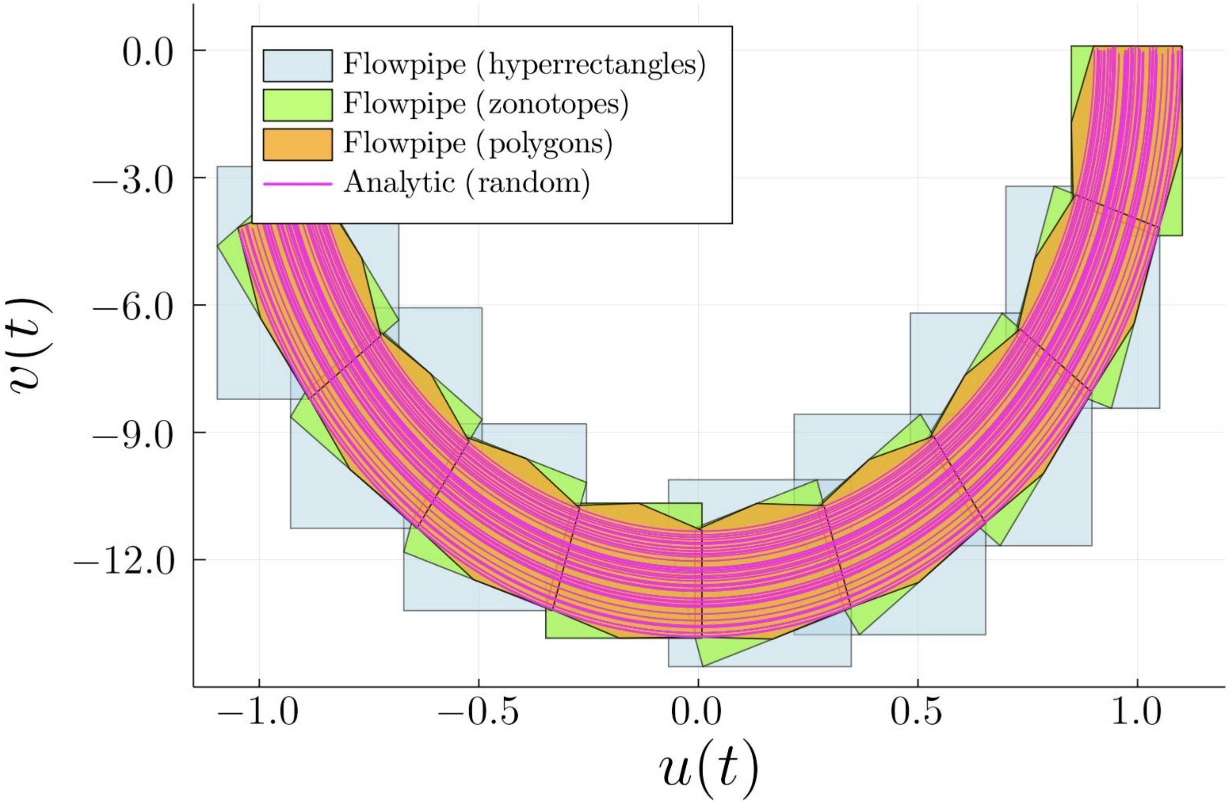
Ejemplo



(b) The flowpipe construction method for Eq. (22) is wrapping-free: the area of the sets does not increase with time.

Forets, Marcelo, Daniel Freire Caporale, and Jorge M. Pérez Zerpa. "Combining set propagation with finite element methods for time integration in transient solid mechanics problems." *Computers & Structures* 259 (2022): 106699.

Combinando reachability y FEM

Problemas de transferencia de calor::

$$\mathbf{C}_\theta \dot{\boldsymbol{\theta}}(t) + \mathbf{K}_\theta \boldsymbol{\theta}(t) = \mathbf{f}_\theta(t)$$

Problemas de dinámica de sólidos:

$$\mathbf{M}_\mathbf{u} \ddot{\mathbf{u}}(t) + \mathbf{C}_\mathbf{u} \dot{\mathbf{u}}(t) + \mathbf{K}_\mathbf{u} \mathbf{u}(t) = \mathbf{f}_\mathbf{u}(t)$$

Forets, Marcelo, Daniel Freire Caporale, and Jorge M. Pérez Zerpa. "Combining set propagation with finite element methods for time integration in transient solid mechanics problems." *Computers & Structures* 259 (2022): 106699.

Combinando reachability y FEM

Algorithm 1: Set propagation method pseudo-code.

Input: $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$: FEM assembled matrices, $\mathbf{F}$: vector of loads, $\mathcal{X}_0 \subseteq R^n$: initial states set
δ: time step increment, N: number of time steps

Output: List of reachable sets

$\mathbf{A}, \mathcal{C}_0 = homogeneize(\mathbf{M}, \mathbf{C}, \mathbf{K}, \mathbf{F})$; `// use Eq.(31) or (32)`
$\Omega_0 = discretize(\mathcal{X}_0 \times \mathcal{C}_0, \mathbf{A}, \delta)$; `// time discretization using Prop.1`
$\mathbf{\Phi} = exp(\mathbf{A}\delta)$; `// matrix exponential`
$X_0 = \Omega_0$; `// reachable states approximation for [0, δ]`
for $k = 0$ **to** $N - 2$ **do**
    $X_{k+1} = \mathbf{\Phi} X_k$; `// set propagation; use either Eq.(36), Prop.2 or Eq.(37)`
end
return $X_0, X_1, \cdots, X_{N-1}$; `// flowpipe approximation`

Forets, Marcelo, Daniel Freire Caporale, and Jorge M. Pérez Zerpa. "Combining set propagation with finite element methods for time integration in transient solid mechanics problems." *Computers & Structures* 259 (2022): 106699.

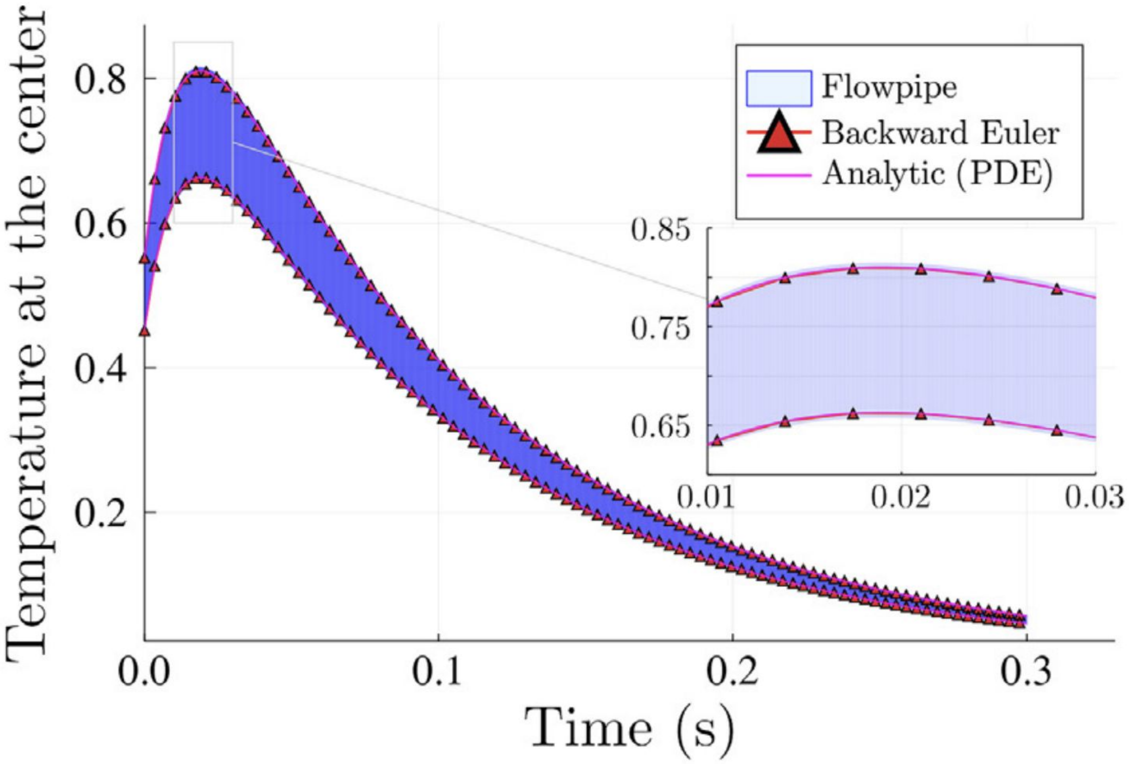
Ejemplo: transferencia de calor 1D

$$\begin{cases} \frac{\partial T(x,t)}{\partial t} = \frac{\kappa}{\rho c} \frac{\partial^2 T(x,t)}{\partial x^2}, & x \in \Omega \\ T(x,t) = 0, & x \in \{0,1\}, \end{cases}$$

$$T(x,0) = (1+\varepsilon)\left(\sin(\pi x) + \frac{1}{2}\sin(3\pi x)\right), \qquad \varepsilon \in [-0.1, 0.1],$$

Forets, Marcelo, Daniel Freire Caporale, and Jorge M. Pérez Zerpa. "Combining set propagation with finite element methods for time integration in transient solid mechanics problems." *Computers & Structures* 259 (2022): 106699.

Ejemplo: transferencia de calor 1D



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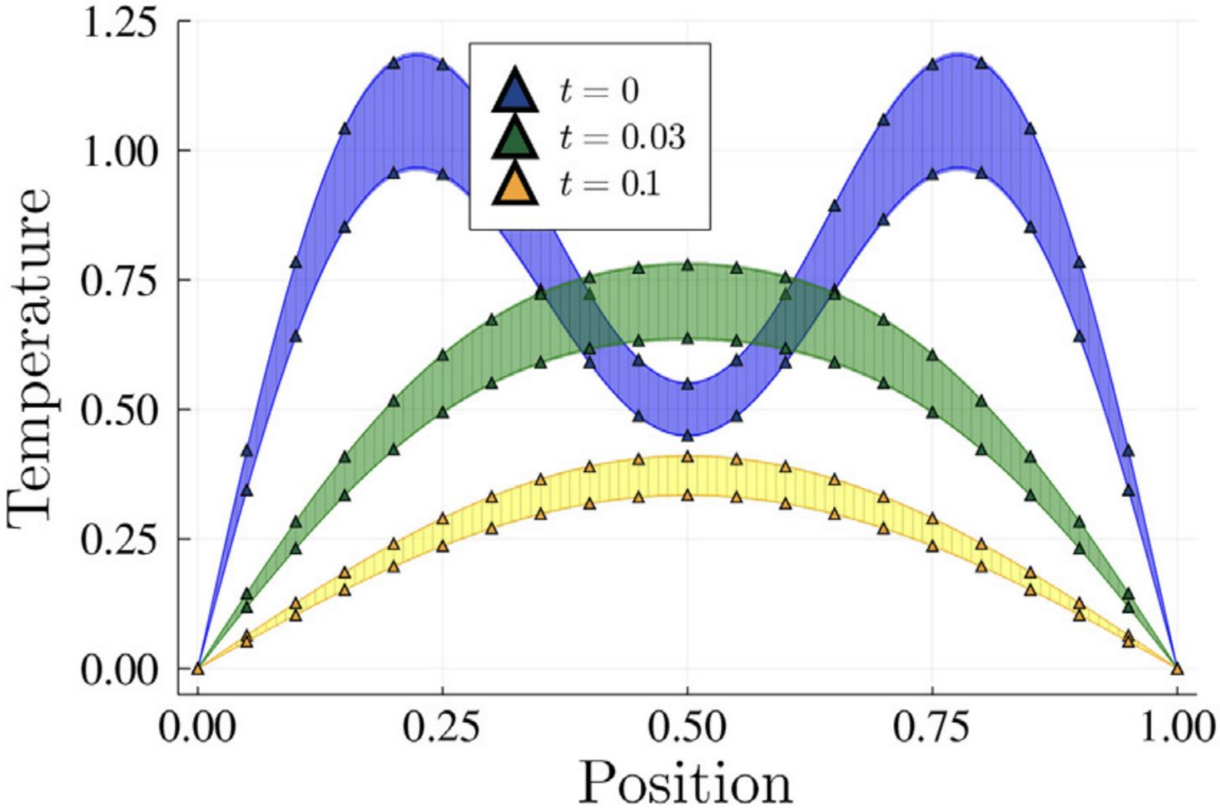

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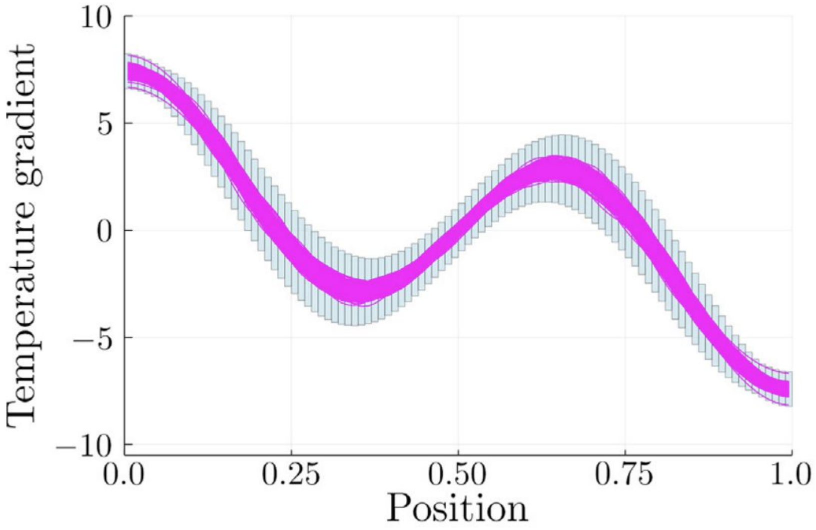
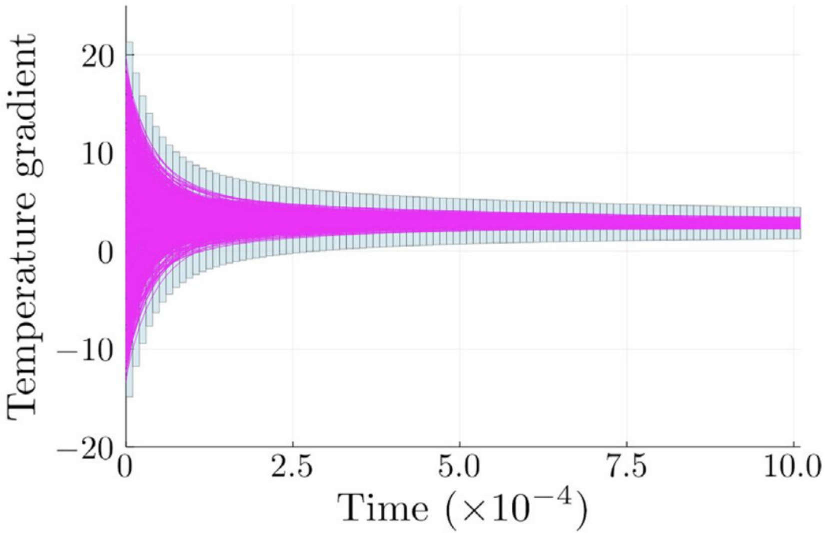


(a) Temperature gradient profile at time $t = 0.001$.



(b) Temperature gradient vs. time at $x = 2/3$.

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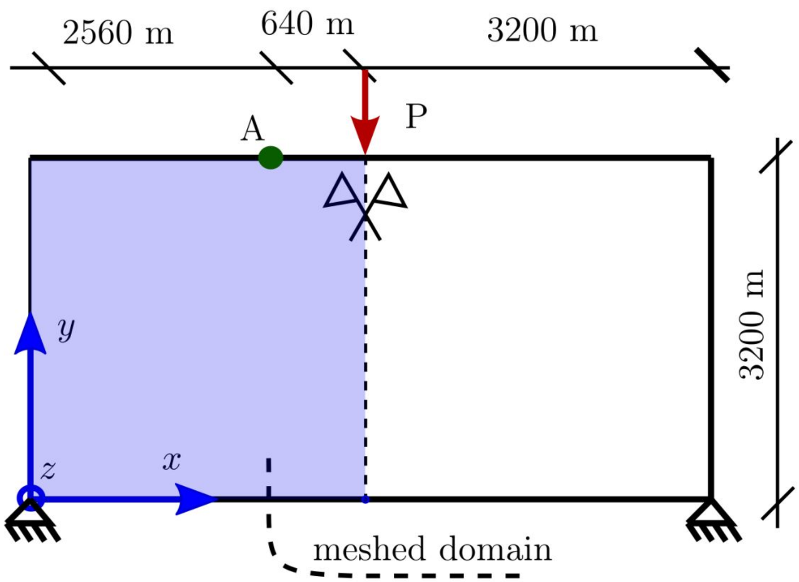
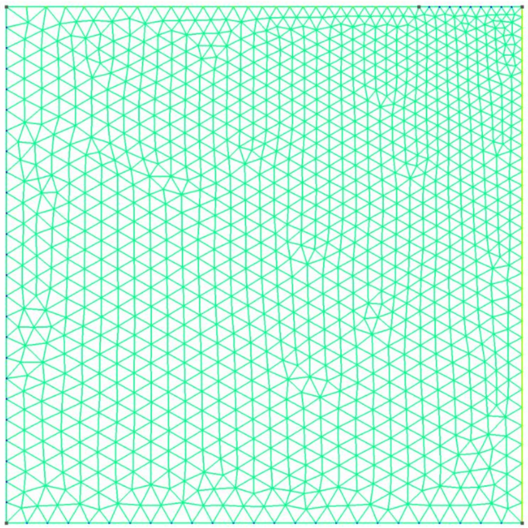
Ejemplo: propagación de ondas 2D



(a) Diagram of domain and boundary conditions considered.



(b) Finite Element Method mesh used, formed by triangular elements.

2235 linear triangular elements

1180 nodes

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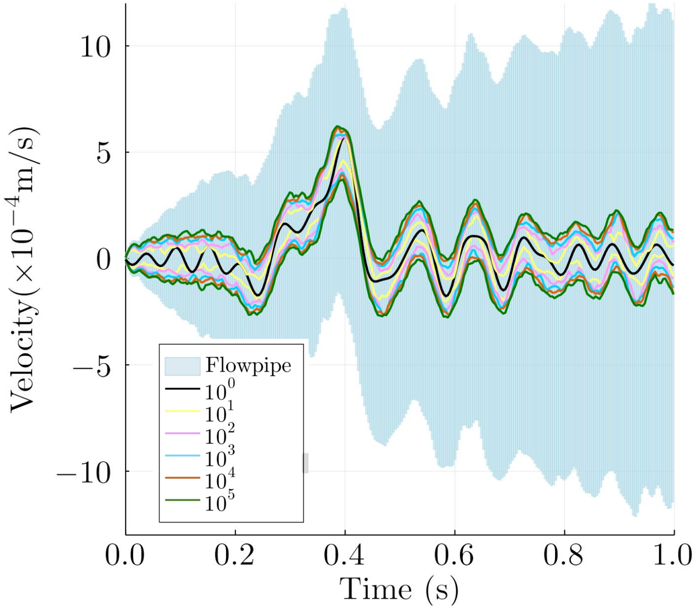
Ejemplo

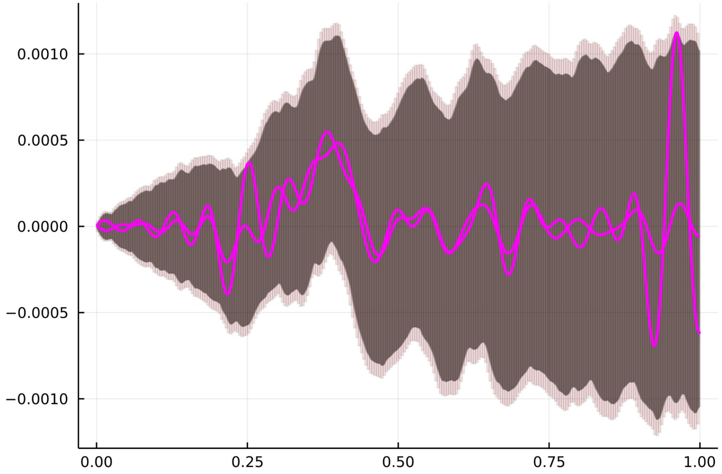


Method	# Trajectories	Time (s)	$\|v_{env}\|_{L_1}$ (10^{-5})	$\|v_{env}\|_{L_\infty}$ (10^{-5})
Newmark	1	0.3	9.27	56.98
Newmark	10	2.0	13.52	57.53
Newmark	100	17.7	16.61	57.59
Newmark	1000	175.5	18.52	58.22
Newmark	10000	1771.4	19.98	61.18
Newmark	100000	17796.1	21.42	62.21
Set Propagation	-	8.5	81.33	122.25

The initial set is propagated exponentially faster than using simulations over randomly sampled initial states

Forets, Marcelo, Daniel Freire Caporale, and Jorge M. Pérez Zerpa. "Combining set propagation with finite element methods for time integration in transient solid mechanics problems." *Computers & Structures* 259 (2022): 106699.

Ejemplo



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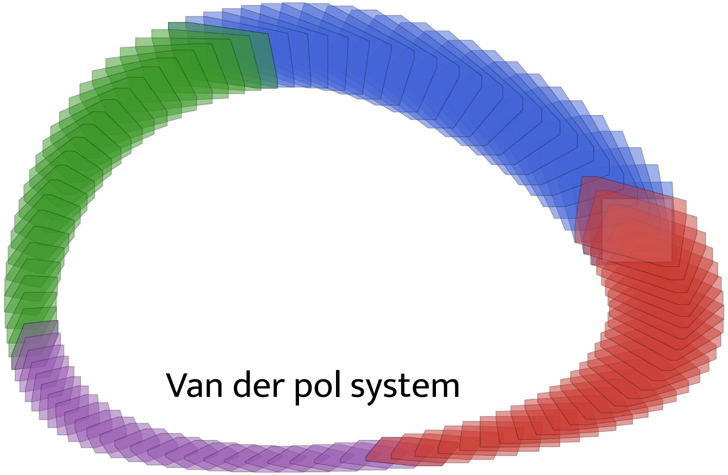
Set Propagation	-	8.5	81.33	122.25

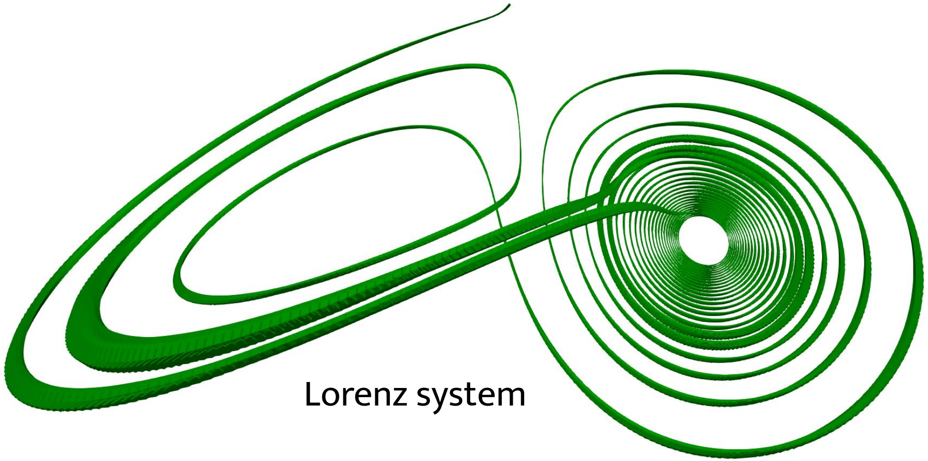
The sequence of sets obtained (flowpipe) converges to the exact reachable states when the time-step decreases

Forets, Marcelo, Daniel Freire Caporale, and Jorge M. Pérez Zerpa. "Combining set propagation with finite element methods for time integration in transient solid mechanics problems." *Computers & Structures* 259 (2022): 106699.

Reachability para sistemas no lineales

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{w}(t)), \qquad \mathbf{x}(0) \in \mathcal{X}_0 \subset \mathbb{R}^n, \quad \mathbf{w}(t) \in \mathcal{W} \subset \mathbb{R}^m.$$





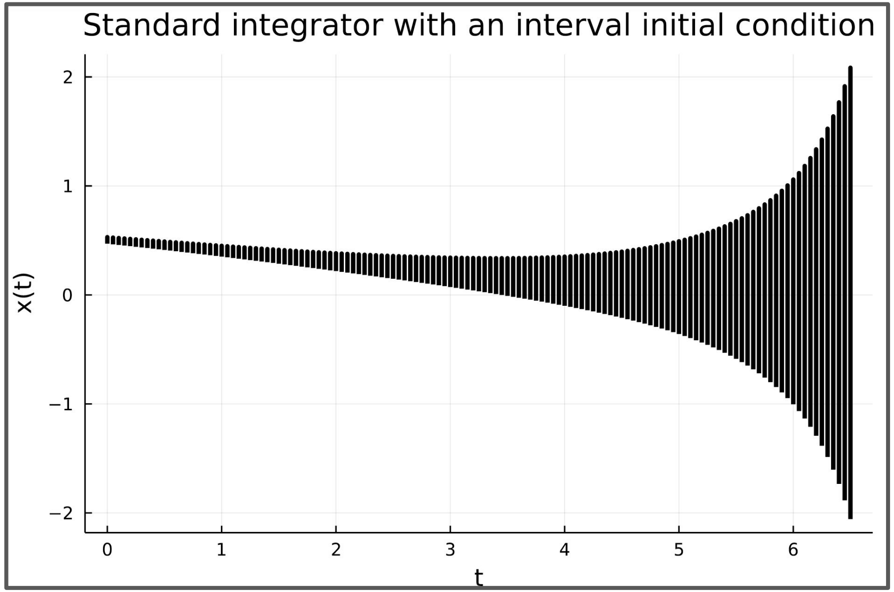
Set-based approaches are classified into: invariant generation, optimization based-approaches, solution-space abstractions, and state-space abstractions.

Ejemplo

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = rx(1 - x/K)$$

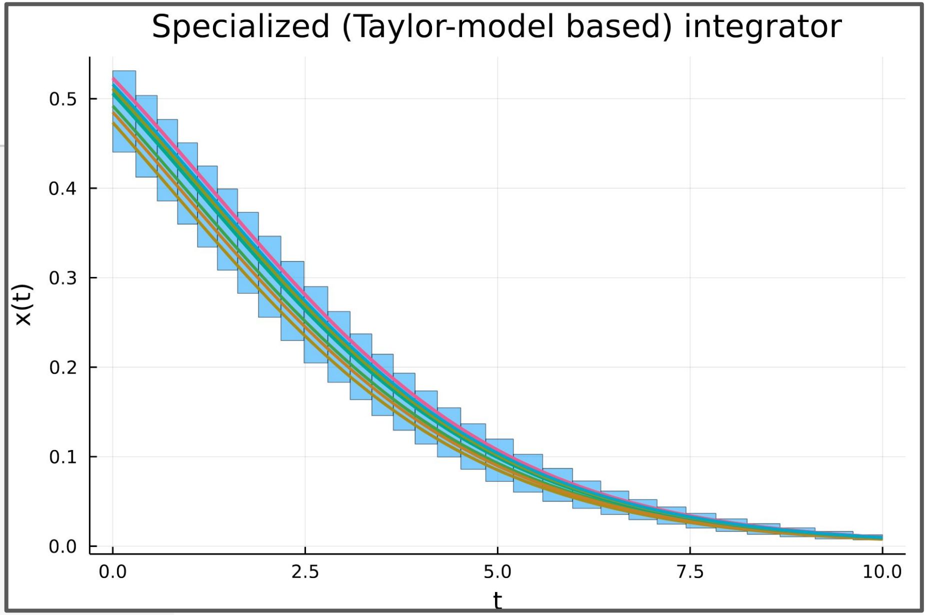
Logistic equation (population dynamics)

```julia
using DifferentialEquations,
      IntervalArithmetic

# define the problem
function f(dx, x, p, t)
    r = -0.5
    K = 0.8
    dx[1] = r*x[1]*(1 - x[1]/K)
end

x₀ = [0.47 .. 0.53]  # initial condition
prob = ODEProblem(f, x₀, (0.0, 6.5))
sol = solve(prob, RK4(), adaptive=false,
            dt=0.05, reltol=1e-6)
```



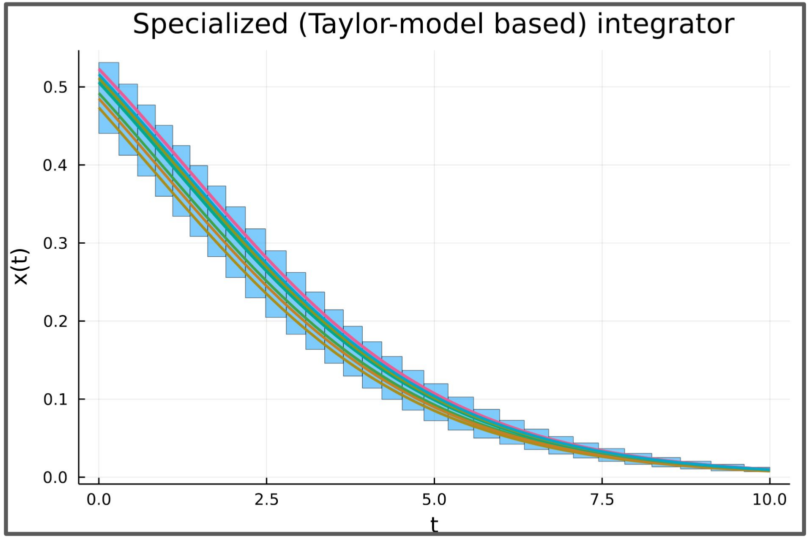
Ejemplo

```julia
using ReachabilityAnalysis, Plots

# define the model (same as before)
function f(dx, x, p, t)
    r = -0.5
    K = 0.8
    dx[1] = r*x[1]*(1 - x[1]/K)
end

X0 = 0.47 .. 0.53 # interval initial states
# initial-value problem
prob = @ivp(x' = f(x), x(0) ∈ X0, dim=1)

# solve it
sol = solve(prob, alg=TMJets21a(abstol=1e-10), T=10.0);
```



Ejemplo

Example:
Taylor model reach-set at the final computed interval




```
1 tspan(sol[end])
```

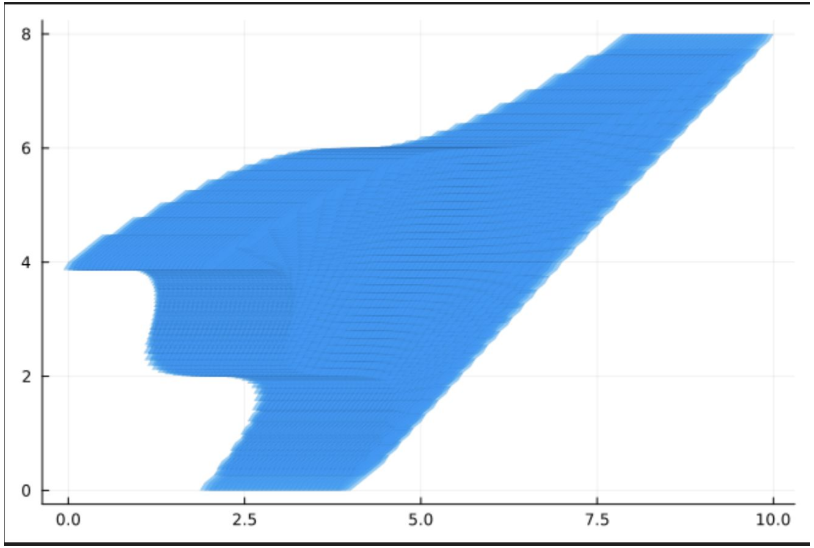
```
[9.61803, 10]
```

```
1 sol[end]
```

```
TaylorModelReachSet{Float64}(TaylorModels.TaylorModel1{TaylorN{Float64}, Float64}[  0.010728640000425796 + 0.0016935
616569102134 x₁ + 0.0001657222430356068 x₁² + ( - 0.0052923801775511875 - 0.0008240688117857138 x₁ - 7.8846059232215
5e-5 x₁²) t + ( 0.0012876075183438367 + 0.00019488963434660352 x₁ + 1.7762397657259186e-5 x₁²) t² + ( - 0.0002030100
35766464 - 2.888459368553032e-5 x₁ - 2.339217437712025e-6 x₁²) t³ + ( 2.2566088815570262e-5 + 2.752383693482914e-6 x
₁ + 1.474683203606484e-7 x₁²) t⁴ + ( - 1.7202398083315093e-6 - 1.1552580809843272e-7 x₁ + 1.1261821093858127e-8 x₁²)
t⁵ + ( 6.016969171459992e-8 - 1.413785130768101e-8 x₁ - 4.570779684951191e-9 x₁²) t⁶ + ( 6.311540762007901e-9 + 3.82
8591659395892e-9 x₁ + 7.063749944165518e-10 x₁²) t⁷ + ( - 1.4955436168686581e-9 - 4.920337196748677e-10 x₁ - 6.83032
0243727477e-11 x₁²) t⁸ + [-1.13178e-10, 1.0917e-10]], [9.61803, 10])
```

Polynomial zonotopes (non-convex)

$$\mathcal{PZ} = \left\{ \sum_{i=1}^{h} \left(\prod_{k=1}^{p} \alpha_k^{E_{(k,i)}} \right) G_{(\cdot,i)} + \sum_{j=1}^{q} \beta_j G_{I(\cdot,j)} \;\middle|\; \alpha_k, \beta_j \in [-1,1] \right\}.$$



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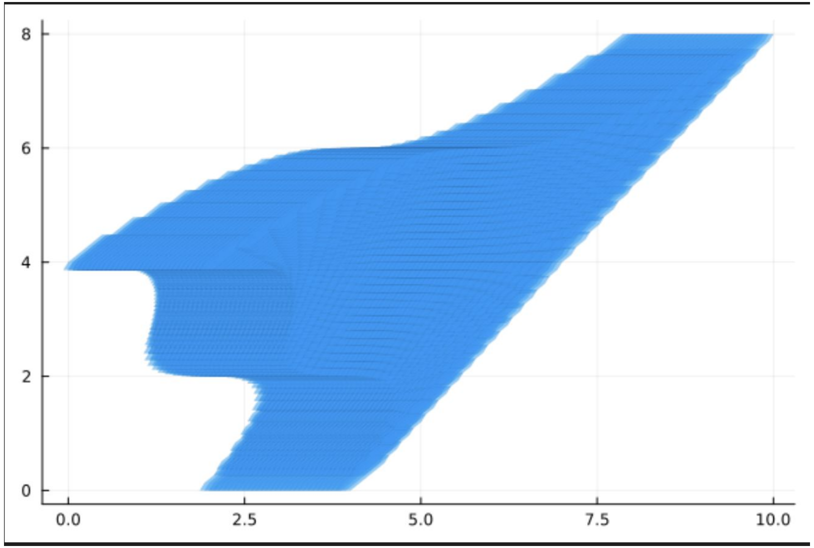


Example 1: *The SPZ*

$$\mathcal{PZ} = \left\langle \begin{bmatrix} 4 & 2 & 1 & 2 \\ 4 & 0 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \end{bmatrix} \right\rangle_{PZ}$$

defines the set

$$\mathcal{PZ} = \left\{ \begin{bmatrix} 4 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \alpha_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \alpha_2 + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \alpha_1^3 \alpha_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \beta_1 \;\middle|\; \alpha_1, \alpha_2, \beta_1 \in [-1,1] \right\}.$$

Reachability mediante abstracción en el espacio de estados

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{w}(t)), \qquad \mathbf{x}(0) \in \mathcal{X}_0 \subset \mathbb{R}^n, \quad \mathbf{w}(t) \in \mathcal{W} \subset \mathbb{R}^m$$

$$\mathbf{z} = [\mathbf{x}^{\mathrm{T}}, \mathbf{w}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{o} \ (o = n + m) \qquad \nabla = \sum_{i=1}^{o} e^{(i)} \frac{\partial}{\partial \mathbf{z}_i}$$

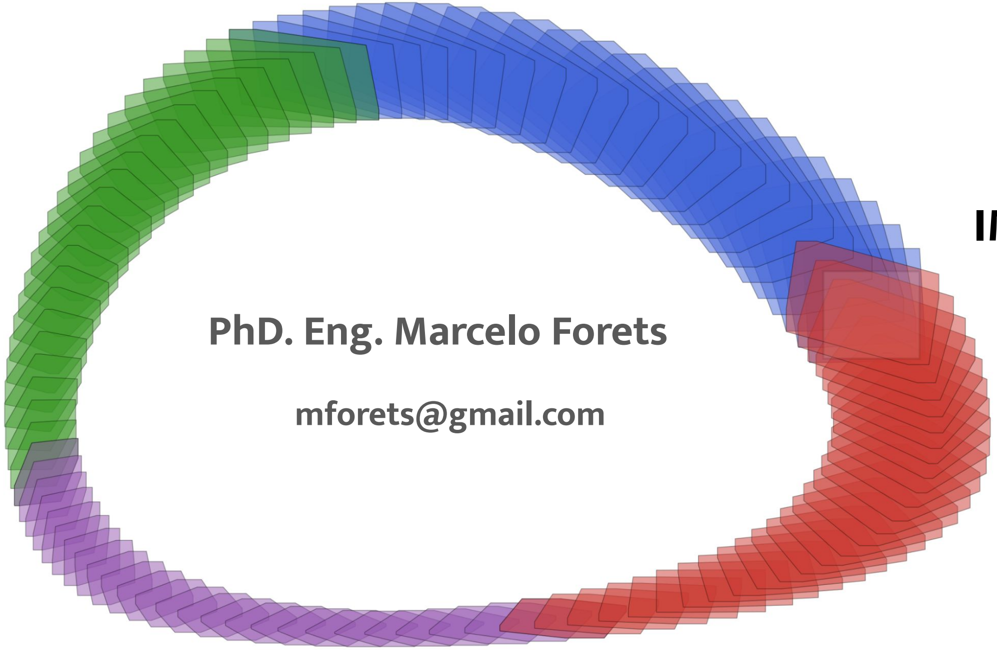
$$\mathcal{L}_i(t) = \left\{ \frac{\left((\mathbf{z}(t) - z^*)^{\mathrm{T}} \nabla\right)^{\kappa+1} f_i(\tilde{\mathbf{z}}(t))}{(\kappa+1)!} \middle| \tilde{\mathbf{z}}(t) = z^* + \alpha(\mathbf{z}(t) - z^*), \alpha \in [0,1] \right\}$$

$$\boxed{\dot{\mathbf{x}}_i(t) \in P^{\kappa}(\mathbf{z}(t), z^*) \oplus \mathcal{L}_i(t)}$$

Ver Kochdumper, N., 2022. *Extensions of polynomial zonotopes and their application to verification of cyber-physical systems* (Doctoral dissertation, Technische Universität München).

Material disponible en: https://github.com/mforets/PresentacionIMERL

¡Gracias por su atención!



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